A subtraction scheme for computing QCD jet cross sections at NNLO: regularization of real-virtual emission

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ABSTRACT: We present a generalization of the dipole subtraction scheme for computing jet cross sections in electron-positron annihilation at next-to-next-to-leading order accuracy in perturbative QCD. In this second part we deal with the regularization of the real-virtual contribution to the NNLO correction.

Contents

1. Introduction

In recent years a lot of effort has been devoted to the extension of the subtraction method of computing QCD corrections at the next-to-leading order (NLO) accuracy to the computation of the radiative corrections at the next-to-next-to-leading order (NNLO) [1–9]. In particular, in ref. [10], a dipole subtraction scheme was defined for computing NNLO

corrections to QCD jet cross sections to processes without coloured partons in the initial state and arbitrary number of massless particles (coloured or colourless) in the final state. That scheme can be summarized as follows.

The NNLO correction to any m-jet cross section is a sum of three contributions, the doubly-real, the one-loop singly-unresolved real-virtual and the two-loop doubly-virtual terms,

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_m \mathrm{d}\sigma^{\mathrm{VV}}_m J_m \,. \tag{1.1}$$

Here the notation for the integrals indicate that the doubly-real corrections involve the fully-differential cross section $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$ of $m+2$ final-state partons, the real-virtual contribution involves the fully-differential cross section for the production of $m+1$ final-state partons at one-loop and the doubly-virtual term is an integral of the fully-differential cross section for the production of m final-state partons at two-loops over the phase space of m partons. The phase spaces are restricted by the corresponding jet functions J_n that define the physical quantity.

In $d = 4$ dimensions the three contributions in eq. (1.1) are separately divergent, but their sum is finite for infrared-safe observables. (The requirement of infrared safety implies certain analytic properties of the jet functions J_n that are spelled out in ref. [8].) As explained in ref. [10] we first continue analitically all integrals to $d = 4 - 2\varepsilon$ dimensions and then rewrite eq. (1.1) as

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{NNLO}}_{m+1} + \int_m \mathrm{d}\sigma^{\mathrm{NNLO}}_m \,, \tag{1.2}$$

that is a sum of integrals,

$$\mathrm{d}\sigma^{\mathrm{NNLO}}_{m+2} = \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}_{\varepsilon=0}, \tag{1.3}$$

$$\mathrm{d}\sigma^{\mathrm{NNLO}}_{m+1} = \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}_{\varepsilon=0}, \tag{1.4}$$

and

$$\mathrm{d}\sigma^{\mathrm{NNLO}}_m = \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_m + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\}_{\varepsilon=0} J_m \,, \tag{1.5}$$

each integrable in four dimensions by construction. Here $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ are approximate cross sections that regularize the doubly-real emission cross section in the one- and two-parton infrared regions of the phase space, respectively. The double subtraction due to the overlap of these two terms is compensated by $\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}$. These terms are defined in ref. [10] explicitly, where the finiteness of $\mathrm{d}\sigma^{\mathrm{NNLO}}_{m+2}$ is demonstrated also numerically for the case of $e^+e^- \to 3$ jets ($m = 3$). We point out that the notation in eqs. (1.3) and (1.4) is symbolic in the sense that each subtraction term is actually a sum of different terms. The jet functions in these terms depend on different sets of momenta. These sets of momenta for the jet function J_m in eq. (1.4) will be presented in this paper.

In ref. [11], we computed the integral $\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ and showed that the terms in the first bracket in eq. (1.4) do not contain ε poles. Nevertheless, those terms still lead to divergent

integrals due to kinematical singularities in the one-parton unresolved parts of the phase space. In this paper we define explicitly $d\sigma_{m+1}^{RV,A_1}$ and $\left(\int_1 d\sigma_{m+2}^{RR,A_1}\right)^{A_1}$, that regularize the singly-unresolved limits of the real-vitrual cross section and $\int_1 d\sigma_{m+2}^{RR,A_1}$ in turn. Thus we complete the presention of all formulae relevant for constructing $d\sigma_{m+1}^{NNLO}$ explicitly.

2. Notation

2.1 Matrix elements

We consider processes with coloured particles (partons) in the final states, while the initial-state particles are colourless (tipically electron-positron annihilation into hadrons). Any number of additional non-coloured final-state particles is allowed, too, but they will be suppressed in the notation. Resolved partons in the final state are labelled by $i, k, l, \ldots$, the unresolved one is denoted by r.

We adopt the colour- and spin-state notation of ref. [12]. In this notation the amplitude for a scattering process involving the final-state momenta $\{p\}$, $|\mathcal{M}_m(\{p\})\rangle$, is an abstract vector in colour and spin space, and its normalization is fixed such that the squared amplitude summed over colours and spins is

$$|\mathcal{M}_m|^2 = \langle \mathcal{M}_m || \mathcal{M}_m \rangle. \tag{2.1}$$

This matrix element has the following formal loop expansion:

$$|\mathcal{M}\rangle = |\mathcal{M}^{(0)}\rangle + |\mathcal{M}^{(1)}\rangle + \ldots, \tag{2.2}$$

where $|\mathcal{M}^{(0)}\rangle$ denotes the tree-level contribution, $|\mathcal{M}^{(1)}\rangle$ is the one-loop contribution and the dots stand for higher-loop contributions, which are not used in this paper.

Colour interactions at the QCD vertices are represented by associating colour charges $\boldsymbol{T}_i$ with the emission of a gluon from each parton i. In the colour-state notation, each vector $|\mathcal{M}\rangle$ is a colour-singlet state, so colour conservation is simply

$$\left(\sum_j \boldsymbol{T}_j\right) |\mathcal{M}\rangle = 0, \tag{2.3}$$

where the sum over j extends over all the external partons of the state vector $|\mathcal{M}\rangle$, and the equation is valid order by order in the loop expansion of eq. (2.2).

Using the colour-state notation, we define the two-parton colour-correlated squared tree amplitudes as

$$|\mathcal{M}_{(i,k)}^{(0)}(\{p\})|^2 \equiv \langle \mathcal{M}^{(0)}(\{p\})| \boldsymbol{T}_i \cdot \boldsymbol{T}_k |\mathcal{M}^{(0)}(\{p\})\rangle \tag{2.4}$$

and similarly the three-parton colour-correlated squared tree amplitudes, $|\mathcal{M}_{(i,k,l)}^{(0)}|^2$ for i, k and l being different, and the doubly two-parton colour-correlated squared tree amplitudes $|\mathcal{M}_{(i,k),(j,l)}^{(0)}|^2$:

$$|\mathcal{M}_{(i,k,l)}^{(0)}|^2 \equiv \sum_{a,b,c} f_{abc} \langle \mathcal{M}^{(0)}|T_i^a T_k^b T_l^c|\mathcal{M}^{(0)}\rangle \tag{2.5}$$

and

$$|\mathcal{M}^{(0)}_{(i,k)(j,l)}|^2 \equiv \langle\mathcal{M}^{(0)}|\left\{\boldsymbol{T}_i\cdot\boldsymbol{T}_k,\boldsymbol{T}_j\cdot\boldsymbol{T}_l\right\}|\mathcal{M}^{(0)}\rangle\,, \tag{2.6}$$

where the anticommutator $\{\boldsymbol{T}_i\cdot\boldsymbol{T}_k,\boldsymbol{T}_j\cdot\boldsymbol{T}_l\}$ is non-trivial only if $i=j$ or $k=l$, see eq. (2.8). We shall also use the two-parton colour-correlated one-loop amplitude, defined using an analogous notation:

$$2\,\mathrm{Re}\langle\mathcal{M}^{(0)}||\mathcal{M}^{(1)}\rangle_{(i,k)} \equiv 2\,\mathrm{Re}\langle\mathcal{M}^{(0)}|\,\boldsymbol{T}_i\cdot\boldsymbol{T}_k\,|\mathcal{M}^{(1)}\rangle\,. \tag{2.7}$$

The colour-charge algebra for the product $(\boldsymbol{T}_i)^n(\boldsymbol{T}_k)^n \equiv \boldsymbol{T}_i\cdot\boldsymbol{T}_k$ is:

$$\boldsymbol{T}_i\cdot\boldsymbol{T}_k = \boldsymbol{T}_k\cdot\boldsymbol{T}_i \quad\text{if}\quad i\neq k; \qquad \boldsymbol{T}_i^2 = C_i\,. \tag{2.8}$$

Here C_i is the eigenvalue of the quadratic Casimir operator in the representation of particle i and we have $C_\mathrm{F} = T_\mathrm{R}(N_\mathrm{c}^2-1)/N_\mathrm{c} = (N_\mathrm{c}^2-1)/(2N_\mathrm{c})$ in the fundamental and $C_\mathrm{A} = 2T_\mathrm{R}N_\mathrm{c} = N_\mathrm{c}$ in the adjoint representation, i.e. we are using the customary normalization $T_\mathrm{R} = 1/2$.

2.2 Dimensional regularization, one-loop amplitudes and renormalization

We employ conventional dimensional regularization (CDR) in $d = 4-2\varepsilon$ space-time dimensions to regulate both the IR and UV divergences, when quarks (spin-$\frac{1}{2}$ Dirac fermions) possess 2 spin polarizations, gluons have $d-2$ helicity states and all particle momenta are taken as d-dimensional.

Turning to the renormalization of the amplitudes, let the perturbative expansion of the scattering amplitude $\mathcal{A}_m$ in terms of the bare coupling $g_\mathrm{s} \equiv \sqrt{4\pi\alpha_\mathrm{s}^\mathrm{u}}$ be

$$|\mathcal{A}_m\rangle = \left(\frac{\alpha_\mathrm{s}^\mathrm{u}\mu^{2\varepsilon}}{4\pi}\right)^{q/2}\left[|\mathcal{A}_m^{(0)}\rangle + \left(\frac{\alpha_\mathrm{s}^\mathrm{u}\mu^{2\varepsilon}}{4\pi}\right)|\mathcal{A}_m^{(1)}\rangle + \mathrm{O}\left((\alpha_\mathrm{s}^\mathrm{u})^2\right)\right]\,, \tag{2.9}$$

where q is a non-negative integer and μ is the dimensional-regularization scale. For the renormalized amplitudes (in the CDR scheme) we use the notation $|\mathcal{M}_m\rangle$. These are obtained from the unrenormalized amplitudes by expressing the bare coupling in terms of the running coupling $\alpha_\mathrm{s}(\mu_R^2)$ evaluated at the arbitrary renormalization scale μ_R^2 as

$$\alpha_\mathrm{s}^\mathrm{u}\mu^{2\varepsilon} = \alpha_\mathrm{s}(\mu_R^2)\,\mu_R^{2\varepsilon}\,S_\varepsilon^{-1}\left[1 - \left(\frac{\alpha_\mathrm{s}(\mu_R^2)}{4\pi}\right)\frac{\beta_0}{\varepsilon} + \mathrm{O}(\alpha_\mathrm{s}^2(\mu_R^2))\right]\,, \tag{2.10}$$

where β_0 is the first coefficient of the β function for n_f number of light quark flavours,

$$\beta_0 = \frac{11}{3}C_\mathrm{A} - \frac{4}{3}T_\mathrm{R}n_\mathrm{f} - \frac{2}{3}T_\mathrm{R}n_\mathrm{s}\,, \tag{2.11}$$

for QCD $n_\mathrm{s} = 0$. In eq. (2.10), S_ε is the phase space factor due to the integral over the $(d-3)$-dimensional solid angle, which is included in the definition of the running coupling

in the $\overline{\text{MS}}$ renormalization scheme,[1]

$$S_\varepsilon = \int \frac{\mathrm{d}^{(d-3)}\Omega}{(2\pi)^{d-3}} = \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)} \,. \tag{2.12}$$

We always consider the running coupling in the $\overline{\text{MS}}$ scheme defined with the inclusion of this phase space factor.

The relations between the renormalized amplitudes of eq. (2.2) and the unrenormalized ones are given as follows:

$$|\mathcal{M}_m^{(0)}\rangle = \left(\frac{\alpha_{\mathrm{s}}(\mu_R^2)\,\mu_R^{2\varepsilon}}{4\pi} S_\varepsilon^{-1} \right)^{q/2} |\mathcal{A}_m^{(0)}\rangle \,, \tag{2.13}$$

$$|\mathcal{M}_m^{(1)}\rangle = \left(\frac{\alpha_{\mathrm{s}}(\mu_R^2)\,\mu_R^{2\varepsilon}}{4\pi} S_\varepsilon^{-1} \right)^{q/2} \frac{\alpha_{\mathrm{s}}(\mu_R^2)}{4\pi} S_\varepsilon^{-1} \left(\mu_R^{2\varepsilon} |\mathcal{A}_m^{(1)}\rangle - \frac{q}{2} \frac{\beta_0}{\varepsilon} S_\varepsilon |\mathcal{A}_m^{(0)}\rangle \right) \,. \tag{2.14}$$

After UV renormalization, the dependence on μ turns into a dependence on μ_R, so physical cross sections depend only on the renormalization scale μ_R. To avoid a cumbersome notation, we therefore set $\mu_R = \mu$ in the rest of the paper. Furthermore, after the IR poles are canceled in an NLO, or NNLO computation, we may set $\varepsilon = 0$, therefore, the $\mu_R^{2\varepsilon}$ and S_ε^{-1} factors that accompany the running coupling in the renormalized amplitude do not give any contribution, so we may perform the substitution

$$\left(\frac{\alpha_{\mathrm{s}}(\mu_R^2)\,\mu_R^{2\varepsilon}}{4\pi} S_\varepsilon^{-1} \right)^{q/2} \left(\frac{\alpha_{\mathrm{s}}(\mu_R^2)}{4\pi} S_\varepsilon^{-1} \right)^{i} \to \left(\frac{\alpha_{\mathrm{s}}(\mu_R^2)}{4\pi} \right)^{q/2+i} \tag{2.15}$$

in eqs. (2.13) and (2.14).

2.3 Remark on regularization-scheme dependence

Although the application of conventional dimensional regularization (CDR) is conceptually clean, the computation of squared matrix elements is much simpler in other versions of dimensional regularization, most notably in dimensional reduction (DR). As a result, most of the multiparton QCD amplitudes $|\mathcal{A}_m^{(n)}\rangle$, both at tree-level ($n = 0$) and one-loop ($n = 1$), are available in DR. At the level of cross sections however, the CDR scheme is used traditionally, therefore, the relation between the two schemes has to be established.

The regularization-scheme (RS) dependence of the matrix elements at tree level affect only terms of $\mathrm{O}(\varepsilon)$, therefore, in computing the $(m + 2)$-parton cross section in eq. (1.3) the RS dependence is completely harmless, the difference vanishes when we take the four-dimensional limit. The subtraction terms that regularize the real emission also depend on the RS. While this dependence does not influence $\mathrm{d}\sigma_{m+2}^{\mathrm{NNLO}}$, it leads to differences (even in divergent terms) when the subtraction terms are integrated over the factorized phase

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[1]The $\overline{\text{MS}}$ renormalization scheme as often employed in the literature uses $S_\varepsilon = (4\pi)^\varepsilon \mathrm{e}^{-\varepsilon\gamma_E}$. It is not difficult to check that the two definitions lead to the same expressions in a computation at the NLO accuracy. At NNLO these lead to slightly different bookkeeping of the IR and UV poles at intermediate steps of the computation, but the physical cross section of infrared-safe observables is the same. Our definition leads to somewhat simpler bookkeeping at the NNLO level.

space of the unresolved parton(s). The standard practice in the literature is to set up the subtraction scheme in CDR and transform the loop matrix elements to CDR if those were obtained in other schemes, for instance, in DR.

The RS dependence in the loop amplitudes has in general both ultraviolet (UV) and infrared (IR) origin. Both have been discussed thoroughly up to two loops in ref. [13]. In the present paper we deal only with one-loop amplitudes and we summarize the transition rules from DR to CDR here.

The UV part of the RS dependence is due to the RS dependence of the renormalization procedure. At the one-loop level it means that eq. (2.14) remains valid, with the same expansion parameter, no matter in which RS the bare amplitudes are computed if we perform the substitution [12]

$$\beta_0 \to \beta_0 + \varepsilon \tilde{\beta}_0^{\mathrm{RS}} \tag{2.16}$$

in eqs. (2.10) and (2.14). By definition in CDR $\tilde{\beta}_0^{\mathrm{CDR}} = 0$. If the bare amplitudes are computed in the DR scheme, then $\tilde{\beta}_0^{\mathrm{DR}} = -C_{\mathrm{A}}/6$.

The IR part of the RS dependence can be decomposed into universal finite terms and non-universal contributions at $\mathrm{O}(\varepsilon)$ [14]. The finite terms are completely factorized,

$$|\mathcal{A}_{m,\mathrm{RS}}^{(1)\mathrm{fin}}(\mu^2; \{p\})\rangle = \frac{1}{2} \left(\sum_i \tilde{\gamma}_i^{\mathrm{RS}} \right) |\mathcal{A}_{m,\mathrm{RS}}^{(0)}(\mu^2; \{p\})\rangle + |F_m^{(1)}(\mu^2; \{p\})\rangle + \mathrm{O}(\varepsilon) , \tag{2.17}$$

while the $\mathrm{O}(\varepsilon)$ contributions do not contribute to $\mathrm{d}\sigma_{m+1}^{\mathrm{NNLO}}$ in the four-dimensional limit. The transition coefficients that relate the amplitudes in the RS's depend only on the flavour of the external partons and were first computed in ref. [15]. If $\tilde{\gamma}_i^{\mathrm{CDR}} = 0$, as always assumed by definition, then

$$\tilde{\gamma}_q^{\mathrm{DR}} = \frac{C_{\mathrm{F}}}{2} , \qquad \tilde{\gamma}_g^{\mathrm{DR}} = \frac{C_{\mathrm{A}}}{6} . \tag{2.18}$$

2.4 Cross sections

In our notation the real-virtual cross section $\mathrm{d}\sigma_{m+1}^{\mathrm{RV}}$ is given by

$$\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} = \mathrm{d}\phi_{m+1}(\{p\}) \, 2\,\mathrm{Re}\langle \mathcal{M}_{m+1}^{(0)} | \mathcal{M}_{m+1}^{(1)} \rangle , \tag{2.19}$$

where $\mathrm{d}\phi_{m+1}(\{p\})$ is the d-dimensional phase space for $m+1$ outgoing particles with momenta $\{p\} \equiv \{p_1, \ldots, p_{m+1}\}$ and total momentum Q,

$$\mathrm{d}\phi_{m+1}(p_1, \ldots, p_{m+1}; Q) = \left[\prod_{i=1}^{m+1} \frac{\mathrm{d}^d p_i}{(2\pi)^{d-1}} \, \delta_+(p_i^2) \right] (2\pi)^d \delta^{(d)}(p_1 + \cdots + p_{m+1} - Q) . \tag{2.20}$$

The integral of the singly-unresolved approximate cross section for doubly-real emission over the factorized one-parton phase space was computed in ref. [11]

$$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} = \mathrm{d}\sigma_{m+1}^{\mathrm{R}} \otimes \boldsymbol{I}(m, \varepsilon) , \tag{2.21}$$

where $\mathrm{d}\sigma_{m+1}^{\mathrm{R}}$ is the Born-level cross section for the emission of $m+1$ partons and $\boldsymbol{I}(m, \varepsilon)$ is an operator acting on the colour space of the $m+1$ final-state partons. The notation on

the right hand side means that one has to write down the expression for $d\sigma^{\mathrm{R}}_{m+1}$ and then replace the Born-level squared matrix element

$$|\mathcal{M}^{(0)}_{m+1}|^2 = \langle \mathcal{M}^{(0)}_{m+1}|\mathcal{M}^{(0)}_{m+1}\rangle\,, \tag{2.22}$$

by

$$\langle \mathcal{M}^{(0)}_{m+1}|\boldsymbol{I}(m,\varepsilon)|\mathcal{M}^{(0)}_{m+1}\rangle\,. \tag{2.23}$$

The insertion operator $\boldsymbol{I}(m,\varepsilon)$ differs from the $\boldsymbol{I}(\varepsilon)$ operator derived in ref. [12] in non-singular terms as ε tends to zero. Explicitly,

$$\boldsymbol{I}(\{p\};m,\varepsilon) = \frac{\alpha_{\mathrm{s}}}{2\pi}\, S_\varepsilon \left(\frac{\mu^2}{Q^2}\right)^\varepsilon \sum_i \left[\mathrm{C}_i(y_{iQ};m,\varepsilon)\boldsymbol{T}_i^2 + \sum_{k\neq i}\mathrm{S}_{ik}(y_{ik},y_{iQ},y_{kQ};m,\varepsilon)\boldsymbol{T}_i\boldsymbol{T}_k\right] \tag{2.24}$$

where $y_{ik} = s_{ik}/Q^2 \equiv 2p_i{\cdot}p_k/Q^2$, $y_{iQ} = 2p_i{\cdot}Q/Q^2$ and with

$$\mathrm{C}_q = \mathrm{C}_{qg} - \mathrm{CS}\,,$$
$$\mathrm{C}_g = \frac{1}{2}\mathrm{C}_{gg} + n_{\mathrm{f}}\mathrm{C}_{q\bar{q}} - \mathrm{CS}\,. \tag{2.25}$$

Explicit expressions for the functions $\mathrm{C}_{ik}(y_{iQ};m,\varepsilon)$, $\mathrm{S}_{ik}(y_{ik},y_{iQ},y_{kQ};m,\varepsilon)$ and $\mathrm{CS}(m,\varepsilon)$ can be found in ref. [11], where it was shown that the ε poles of the one-loop squared matrix element $2\,\mathrm{Re}\langle\mathcal{M}^{(0)}_{m+1}|\mathcal{M}^{(1)}_{m+1}\rangle$ are cancelled exactly by $\langle\mathcal{M}^{(0)}_{m+1}|\boldsymbol{I}(\{p\};m,\varepsilon)|\mathcal{M}^{(0)}_{m+1}\rangle$.

3. Counterterms for the real-virtual cross section

3.1 Factorization in the collinear and soft limits

In order to devise the approximate cross section $d\sigma^{\mathrm{RV,A_1}}_{m+1}$, we have to study the factorization properties of one-loop squared matrix elements when one parton becomes soft or collinear to another parton. The relevant factorization formulae have been computed in refs. [16 – 21]. In our work we use the formulae of ref. [20] for collinear parton splitting and those in ref. [21] for soft gluon emission. However, the notation in those papers is not convenient for writing factorization formulae which avoid double counting in the soft-collinear limit, therefore, we present new formulae here.

3.1.1 Collinear limit

We define the collinear limit of two final-state momenta p_i and p_r with the help of an auxiliary light-like vector n_{ir}^μ ($n_{ir}^2 = 0$) using the usual Sudakov parametrization,

$$p_i^\mu = z_i p_{ir}^\mu - k_{\perp,r}^\mu - \frac{k_{\perp,r}^2}{z_i}\frac{n_{ir}^\mu}{2p_{ir}n_{ir}}\,, \quad p_r^\mu = z_r p_{ir}^\mu + k_{\perp,r}^\mu - \frac{k_{\perp,r}^2}{z_r}\frac{n_{ir}^\mu}{2p_{ir}n_{ir}}\,, \tag{3.1}$$

where p_{ir}^μ is a light-like momentum that points towards the collinear direction and $k_{\perp,r}$ is the momentum component that is orthogonal to both p_{ir} and n_{ir} ($p_{ir}{\cdot}k_{\perp,r} = n_{ir}{\cdot}k_{\perp,r} = 0$). Momentum conservation requires that $z_i + z_r = 1$. The two-particle invariant mass of the collinear partons is

$$s_{ir} = -\frac{k_{\perp,r}^2}{z_i z_r}\,. \tag{3.2}$$

The collinear limit is defined by the uniform rescaling

$$k_{\perp,r} \to \lambda k_{\perp,r}, \tag{3.3}$$

and taking the limit $\lambda \to 0$, when the one-loop squared matrix element of an $(m+1)$-parton process has the following asymptotic form [20]:[2]

$$2\,\mathrm{Re}\langle\mathcal{M}_{m+1}^{(0)}(p_i, p_r, \ldots)||\mathcal{M}_{m+1}^{(1)}(p_i, p_r, \ldots)\rangle \simeq$$

$$\simeq 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{s_{ir}}\left[2\,\mathrm{Re}\langle\mathcal{M}_m^{(0)}(p_{ir}, \ldots)|\hat{P}_{f_if_r}^{(0)}|\mathcal{M}_m^{(1)}(p_{ir}, \ldots)\rangle\right.$$

$$\left. + 8\pi\alpha_{\mathrm{s}}\,c_\Gamma\left(\frac{\mu^2}{s_{ir}}\right)^\varepsilon\cos(\pi\varepsilon)\,\langle\mathcal{M}_m^{(0)}(p_{ir}, \ldots)|\hat{P}_{f_if_r}^{(1)}|\mathcal{M}_m^{(0)}(p_{ir}, \ldots)\rangle\right] \tag{3.4}$$

where

$$c_\Gamma = \frac{1}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}. \tag{3.5}$$

The meaning of the $\simeq$ sign in eq. (3.4) is that we have neglected subleading terms (in this case those that are less singular than $1/\lambda^2$). In order to simplify further discussion, following the notation of ref. [8], we introduce a symbolic operator $\mathbf{C}_{ir}$ that performs the action of taking the collinear limit of the one-loop squared matrix element, keeping the leading singular term:

$$\mathbf{C}_{ir}\,2\,\mathrm{Re}\langle\mathcal{M}_{m+1}^{(0)}(p_i, p_r, \ldots)||\mathcal{M}_{m+1}^{(1)}(p_i, p_r, \ldots)\rangle =$$

$$= 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{s_{ir}}\left[2\,\mathrm{Re}\langle\mathcal{M}_m^{(0)}(p_{ir}, \ldots)|\hat{P}_{f_if_r}^{(0)}|\mathcal{M}_m^{(1)}(p_{ir}, \ldots)\rangle\right.$$

$$\left. + 8\pi\alpha_{\mathrm{s}}\,c_\Gamma\left(\frac{\mu^2}{s_{ir}}\right)^\varepsilon\cos(\pi\varepsilon)\,\langle\mathcal{M}_m^{(0)}(p_{ir}, \ldots)|\hat{P}_{f_if_r}^{(1)}|\mathcal{M}_m^{(0)}(p_{ir}, \ldots)\rangle\right]. \tag{3.6}$$

The m-parton matrix elements on the right-hand side of eq. (3.4) are obtained from the $(m+1)$-parton matrix elements by removing partons i and r and replacing them with a single parton denoted by ir. The parton ir carries the quantum numbers of the pair $i + r$ in the collinear limit: its momentum is p_{ir}^μ and its other quantum numbers (flavour, colour) are obtained according to the following rule: anything + gluon gives anything and quark + antiquark gives gluon. The kernels $\hat{P}_{f_if_r}^{(0)}$ and $\hat{P}_{f_if_r}^{(1)}$ are the d-dimensional Altarelli-Parisi splitting functions and their one-loop corrections, which depend on the momentum fractions of the decay products and on the relative transverse momentum of the pair. For the sake of simplicity, we label the momentum fractions belonging to a certain parton flavour with the corresponding label of the squared matrix element, $z_{f_i} = z_i$. In the case of splitting into a pair, only one momentum fraction is independent, yet, we find it more convenient to keep the functional dependence on both z_i and z_r. Depending on the f_i

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[2]Since we deal with final-state singularities only, we have $s_{ir} > 0$ and we can write the usual factor $(-\mu^2/s_{ir})^\varepsilon$ as $(\mu^2/s_{ir})^\varepsilon\cos(\pi\varepsilon)$.

flavours of the splitting products the explicit functional forms are[3]

$$\langle\mu|\hat{P}^{(0)}_{g_ig_r}(z_i, z_r, k^{\mu}_{\perp}; \varepsilon)|\nu\rangle = 2C_{\mathrm{A}}\left[-g^{\mu\nu}\left(\frac{z_i}{z_r} + \frac{z_r}{z_i}\right) - 2(1 - \varepsilon)z_iz_r\frac{k^{\mu}_{\perp}k^{\nu}_{\perp}}{k^2_{\perp}}\right], \qquad (3.7)$$

$$\langle\mu|\hat{P}^{(0)}_{\bar{q}_iq_r}(z_i, z_r, k^{\mu}_{\perp}; \varepsilon)|\nu\rangle = T_{\mathrm{R}}\left[-g^{\mu\nu} + 4z_iz_r\frac{k^{\mu}_{\perp}k^{\nu}_{\perp}}{k^2_{\perp}}\right], \qquad (3.8)$$

$$\langle r|\hat{P}^{(0)}_{q_ig_r}(z_i, z_r; \varepsilon)|s\rangle = \delta_{rs}C_{\mathrm{F}}\left[\frac{1 + z^2_i}{z_r} - \varepsilon z_r\right] \equiv \delta_{rs}P^{(0)}_{q_ig_r}(z_i, z_r; \varepsilon), \qquad (3.9)$$

where in the last equation we introduced our notation for the spin-averaged splitting function,

$$P_{f_if_r}(z_i, z_r; \varepsilon) \equiv \langle\hat{P}_{f_if_r}(z_i, z_r, k^{\mu}_{\perp}; \varepsilon)\rangle. \qquad (3.10)$$

The one-loop kernels are

$$\langle\mu|\hat{P}^{(1)}_{g_ig_r}(z_i, z_r, k^{\mu}_{\perp}; \varepsilon)|\nu\rangle = r^{g_ig_r}_{\mathrm{S,ren}}(z_i, z_r; \varepsilon)\langle\mu|\hat{P}^{(0)}_{g_ig_r}(z_i, z_r, k^{\mu}_{\perp}; \varepsilon)|\nu\rangle$$
$$- 4\,C_{\mathrm{A}}\,r^{g_ig_r}_{\mathrm{N.S.}}\left[1 - 2\varepsilon z_iz_r\right]\frac{k^{\mu}_{\perp}k^{\nu}_{\perp}}{k^2_{\perp}}, \qquad (3.11)$$

$$\langle\mu|\hat{P}^{(1)}_{\bar{q}_iq_r}(z_i, z_r, k^{\mu}_{\perp}; \varepsilon)|\nu\rangle = r^{\bar{q}_iq_r}_{\mathrm{S,ren}}(z_i, z_r; \varepsilon)\langle\mu|\hat{P}^{(0)}_{\bar{q}_iq_r}(z_i, z_r, k^{\mu}_{\perp}; \varepsilon)|\nu\rangle, \qquad (3.12)$$

$$\langle r|\hat{P}^{(1)}_{q_ig_r}(z_i, z_r; \varepsilon)|s\rangle = r^{q_ig_r}_{\mathrm{S,ren}}(z_i, z_r; \varepsilon)\langle r|\hat{P}^{(0)}_{q_ig_r}(z_i, z_r; \varepsilon)|s\rangle + \delta_{rs}\,C_{\mathrm{F}}\,r^{q_ig_r}_{\mathrm{N.S.}}\left[1 - \varepsilon z_r\right]. \quad (3.13)$$

The $r_{\mathrm{S,ren}}(z_i, z_r; \varepsilon)$ singular factors are expressed in terms of corresponding unrenormalized $r_{\mathrm{S}}(z_i, z_r; \varepsilon)$ factors. The relation between the two forms is given by the equation:

$$r_{\mathrm{S,ren}}(z_i, z_r; \varepsilon) = r_{\mathrm{S}}(z_i, z_r; \varepsilon) - \frac{\beta_0}{2\varepsilon}\frac{S_{\varepsilon}}{(4\pi)^2c_{\Gamma}}\left[\left(\frac{\mu^2}{s_{ir}}\right)^{\varepsilon}\cos(\pi\varepsilon)\right]^{-1}. \qquad (3.14)$$

The unrenormalized $r^{f_if_r}_{\mathrm{S}}(z_i, z_r; \varepsilon)$ factors and the $r^{f_if_r}_{\mathrm{N.S.}}$ factors can be trivially obtained from the $\mathcal{D}^{\mu, 1-\mathrm{loop}}_{f_{ir}\to f_if_r}$ functions, that were computed in ref. [20]. In the case of gluon splitting those functions are symmetric under the exchange of z_i and z_r. To make this symmetry manifest, we have re-cast the original expression of ref. [20] for the gluon splitting into a $q\bar{q}$ pair into an equivalent form which exhibits the $z_i \leftrightarrow z_r$ symmetry,

$$r^{\bar{q}_iq_r}_{\mathrm{S}}(z_i, z_r; \varepsilon) = \frac{1}{\varepsilon^2}(C_{\mathrm{A}} - 2C_{\mathrm{F}}) + \frac{C_{\mathrm{A}}}{\varepsilon^2}\sum^{\infty}_{m=1}\varepsilon^m\left[\mathrm{Li}_m\left(\frac{-z_i}{z_r}\right) + \mathrm{Li}_m\left(\frac{-z_r}{z_i}\right)\right] \qquad (3.15)$$
$$+ \frac{1}{1 - 2\varepsilon}\left[\frac{1}{\varepsilon}\left(\frac{11}{3}C_{\mathrm{A}} - \frac{4T_{\mathrm{R}}}{3}n_{\mathrm{f}} - \frac{2T_{\mathrm{R}}}{3}n_{\mathrm{s}} - 3C_{\mathrm{F}}\right) + C_{\mathrm{A}} - 2C_{\mathrm{F}} + \frac{C_{\mathrm{A}} + 4T_{\mathrm{R}}(n_{\mathrm{f}} - n_{\mathrm{s}})}{3(3 - 2\varepsilon)}\right].$$

In the case of gluon splitting into two gluons, the same symmetry is valid, which, however, we choose not to make manifest. Instead, we use a form, where the polylogarithms are regular in the $z_r \to 0$ limit (which will be convenient when we compute the soft limit of this expression in section 3.1.3, see eq. (3.24)),

$$r^{g_ig_r}_{\mathrm{S}}(z_i, z_r; \varepsilon) = -\frac{C_{\mathrm{A}}}{\varepsilon^2}\left[\left(\frac{z_i}{z_r}\right)^{\varepsilon}\frac{\pi\varepsilon}{\sin(\pi\varepsilon)} - \sum^{\infty}_{m=1}2\varepsilon^{2m-1}\mathrm{Li}_{2m-1}\left(\frac{-z_r}{z_i}\right)\right], \qquad (3.16)$$

---

[3]We remind the reader that the formulae are valid in the CDR scheme.

and similarly in the case of quark splitting,

$$r_{\mathrm{S}}^{q_i g_r}(z_i, z_r; \varepsilon) = -\frac{1}{\varepsilon^2}\left[C_{\mathrm{A}} \left(\frac{z_i}{z_r}\right)^\varepsilon \frac{\pi\varepsilon}{\sin(\pi\varepsilon)} + \sum_{m=1}^\infty \varepsilon^m \Big[(1 + (-1)^m) C_{\mathrm{A}} - 2 C_{\mathrm{F}} \Big] \mathrm{Li}_m\left(\frac{-z_r}{z_i}\right) \right].$$
(3.17)

Eq. (3.16) also shows that polylogarithms with even subscripts do not appear in the ε expansion of the $r_{\mathrm{S}}^{g_i g_r}$ singular function for gluon splitting.

The $r_{\mathrm{N.S.}}$ non-singular factors do not depend on the momentum fractions,

$$r_{\mathrm{N.S.}}^{g_i g_r} = \frac{C_{\mathrm{A}}(1-\varepsilon) - 2 T_{\mathrm{R}}(n_{\mathrm{f}} - n_{\mathrm{s}})}{(1-2\varepsilon)(2-2\varepsilon)(3-2\varepsilon)}, \qquad r_{\mathrm{N.S.}}^{q_i g_r} = \frac{C_{\mathrm{A}} - C_{\mathrm{F}}}{1 - 2\varepsilon}.$$
(3.18)

In eqs. (3.16) and (3.18) n_{f} and n_{s} denote the number of fundamental fermions and scalars that can circulate in the loops. The case of QCD is obtained by setting $n_{\mathrm{s}} = 0$.

The gluon-gluon and quark-antiquark splittings are symmetric in the momentum fractions of the two decay products (even though this is not manifest for gluon-gluon splitting), while the quark-gluon splitting is not. Nevertheless, we do not distinguish the flavour kernels $\hat{P}_{\mathrm{qg}}$ and $\hat{P}_{\mathrm{gq}}$. The ordering of the flavour indices and arguments of the Altarelli-Parisi kernels has no meaning in our notation, i.e.,

$$\hat{P}_{f_i f_r}(z_i, z_r; \varepsilon) = \hat{P}_{f_r f_i}(z_r, z_i; \varepsilon).$$
(3.19)

Thus, it is sufficient to record the kernel belonging to one ordering. We keep this convention throughout.

3.1.2 Soft limit

The soft limit is defined by parametrizing the soft momentum as $p_r^\mu = \lambda q_r^\mu$ and letting $\lambda \to 0$ at fixed q_r^μ. Neglecting terms less singular than $1/\lambda^2$, it was found [21] that

$$
\begin{aligned}
2\,\mathrm{Re}\langle \mathcal{M}_{m+1}^{(0)}(p_r, \ldots) || \mathcal{M}_{m+1}^{(1)}(p_r, \ldots)\rangle \simeq & \\
-8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon} \sum_i \sum_{k\neq i} \frac{1}{2}\mathcal{S}_{ik}(r) &\bigg\{ 2\,\mathrm{Re}\langle \mathcal{M}_m^{(0)}(\ldots)| \boldsymbol{T}_i \boldsymbol{T}_k | \mathcal{M}_m^{(1)}(\ldots)\rangle \\
-8\pi\alpha_{\mathrm{s}} c_\Gamma \bigg[&\left(\left(C_{\mathrm{A}} \frac{1}{\varepsilon^2} \frac{\pi\varepsilon}{\sin(\pi\varepsilon)} \left(\frac{1}{2}\mu^2 \mathcal{S}_{ik}(r)\right)^\varepsilon \cos(\pi\varepsilon) + \frac{\beta_0}{2\varepsilon}\frac{S_\varepsilon}{(4\pi)^2 c_\Gamma}\right) |\mathcal{M}_{m;(i,k)}^{(0)}(\ldots)|^2 \\
& -2\frac{\pi}{\varepsilon} \sum_{l\neq i,k} \left(\frac{1}{2}\mu^2 \mathcal{S}_{kl}(r)\right)^\varepsilon |\mathcal{M}_{m;(i,k,l)}^{(0)}(\ldots)|^2 \bigg] \bigg\}
\end{aligned}
$$
(3.20)

if r is a gluon. Similarly to the $\boldsymbol{C}_{ir}$ operator of taking the collinear limit, following ref. [8] we introduce another symbolic operator $\boldsymbol{S}_r$ that performs the action of taking the soft limit of the squared matrix element, keeping the leading singular terms. With this notation $\boldsymbol{S}_r\, 2\,\mathrm{Re}\langle \mathcal{M}_{m+1}^{(0)}(p_r, \ldots) || \mathcal{M}_{m+1}^{(1)}(p_r, \ldots)\rangle$ is equal to the right hand side of eq. (3.20) if r is a gluon and $\boldsymbol{S}_r\, 2\,\mathrm{Re}\langle \mathcal{M}_{m+1}^{(0)}(p_r, \ldots) || \mathcal{M}_{m+1}^{(1)}(p_r, \ldots)\rangle = 0$ if r is a fermion.

In eq. (3.20) the m-parton matrix element on the right-hand side is obtained from the $(m+1)$-parton matrix element on the left-hand side by simply removing the soft parton. The eikonal factor is

$$\mathcal{S}_{ik}(r) = \frac{2s_{ik}}{s_{ir}s_{rk}}.$$ (3.21)

Note that eq. (3.20) is valid only for the case of final-state partons. The general case can be found in [21].

3.1.3 Matching the collinear and soft limits

If we want to regularize the squared matrix elements in all singly-unresolved regions of the phase space then we have to subtract all possible collinear and soft limits, i.e. subtract the sum

$$\sum_r \left(\sum_{i \neq r} \frac{1}{2} \mathbf{C}_{ir} + \mathbf{S}_r \right) 2 \operatorname{Re} \langle \mathcal{M}_{m+1}^{(0)}(p_i, p_r, \ldots) || \mathcal{M}_{m+1}^{(1)}(p_i, p_r, \ldots) \rangle,$$ (3.22)

where the $1/2$ symmetry factor appears because in the summation each collinear configuration is taken into account twice. Subtracting eq. (3.22) we perform a double subtraction in some regions of the phase space where the soft and collinear limits overlap. In order to compensate for the double subtraction, we need to find the collinear limit of the right hand side of eq. (3.20) when gluon r becomes simultaneously collinear to parton i. In deriving this limit we use that in the collinear limits (i) the factors multiplying two-parton colour-correlated squared matrix elements are independent of k, therefore using colour conservation (eq. (2.3)) we can perform the summation over k; (ii) the factors multiplying the three-parton colour-correlated squared matrix element are symmetric in k and l while $|\mathcal{M}_{m;(i,k,l)}^{(0)}(p_i, \ldots)|^2$ is antisymmetric, thus the sum of those terms is zero. Finally we have

$$\mathbf{C}_{ir}\mathbf{S}_r \, 2 \operatorname{Re} \langle \mathcal{M}_{m+1}^{(0)}(p_i, p_r, \ldots) || \mathcal{M}_{m+1}^{(1)}(p_i, p_r, \ldots) \rangle = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon} \frac{2}{s_{ir}} \frac{z_i}{z_r} \boldsymbol{T}_i^2$$

$$\times \left[2 \operatorname{Re} \langle \mathcal{M}_m^{(0)}(p_i, \ldots) || \mathcal{M}_m^{(1)}(p_i, \ldots) \rangle \right.$$

$$\left. - 8\pi\alpha_{\mathrm{s}}c_\Gamma \left(C_{\mathrm{A}} \frac{1}{\varepsilon^2} \frac{\pi\varepsilon}{\sin(\pi\varepsilon)} \left(\frac{\mu^2}{s_{ir}} \frac{z_i}{z_r} \right)^\varepsilon \cos(\pi\varepsilon) + \frac{\beta_0}{2\varepsilon} \frac{S_\varepsilon}{(4\pi)^2 c_\Gamma} \right) |\mathcal{M}_m^{(0)}(p_i, \ldots)|^2 \right].$$ (3.23)

Similarly, the soft limit of eq. (3.6) when r is a gluon and $z_r \to 0$ is

$$\mathbf{S}_r\mathbf{C}_{ir} \, 2 \operatorname{Re} \langle \mathcal{M}_{m+1}^{(0)}(p_i, p_r, \ldots) || \mathcal{M}_{m+1}^{(1)}(p_i, p_r, \ldots) \rangle = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon} \frac{2}{s_{ir}} \frac{1}{z_r} \boldsymbol{T}_i^2$$

$$\times \left[2 \operatorname{Re} \langle \mathcal{M}_m^{(0)}(p_i, \ldots) || \mathcal{M}_m^{(1)}(p_i, \ldots) \rangle \right.$$

$$\left. - 8\pi\alpha_{\mathrm{s}}c_\Gamma \left(C_{\mathrm{A}} \frac{1}{\varepsilon^2} \frac{\pi\varepsilon}{\sin(\pi\varepsilon)} \left(\frac{\mu^2}{s_{ir}} \frac{1}{z_r} \right)^\varepsilon \cos(\pi\varepsilon) + \frac{\beta_0}{2\varepsilon} \frac{S_\varepsilon}{(4\pi)^2 c_\Gamma} \right) |\mathcal{M}_m^{(0)}(p_i, \ldots)|^2 \right].$$ (3.24)

Eqs. (3.23) and (3.24) differ by the term $z_i = 1 - z_r$ in the numerator of eq. (3.23), which is subleading if r is soft. Therefore, eq. (3.23) can be used to account for the double

subtraction: it cancels the soft subtraction in the collinear limit by construction,

$$\mathbf{C}_{ir}(\mathbf{S}_r - \mathbf{C}_{ir}\mathbf{S}_r)2\,\mathrm{Re}\langle\mathcal{M}_{m+1}^{(0)}||\mathcal{M}_{m+1}^{(1)}\rangle = 0\,, \tag{3.25}$$

and the $\mathbf{C}_{ir} - \mathbf{C}_{ir}\mathbf{S}_r$ difference is subleading in the soft limit,

$$\mathbf{S}_r(\mathbf{C}_{ir} - \mathbf{C}_{ir}\mathbf{S}_r)2\,\mathrm{Re}\langle\mathcal{M}_{m+1}^{(0)}||\mathcal{M}_{m+1}^{(1)}\rangle = 0\,. \tag{3.26}$$

Accordingly, in order to remove the double subtraction from eq. (3.22), we have to add terms like that in eq. (3.23). That amounts to always take the collinear limit of the soft factorization formula rather than the reverse (like terms in eq. (3.24)). Thus the candidate for a subtraction term for regularizing the squared matrix element in all singly-unresolved limits is

$$\mathbf{A}_1\,2\,\mathrm{Re}\langle\mathcal{M}_{m+1}^{(0)}||\mathcal{M}_{m+1}^{(1)}\rangle = \tag{3.27}$$
$$\sum_r\left[\sum_{i\neq r}\frac{1}{2}\mathbf{C}_{ir} + \left(\mathbf{S}_r - \sum_{i\neq r}\mathbf{C}_{ir}\mathbf{S}_r\right)\right]2\,\mathrm{Re}\langle\mathcal{M}_{m+1}^{(0)}(p_i,p_r,\ldots)||\mathcal{M}_{m+1}^{(1)}(p_i,p_r,\ldots)\rangle\,.$$

Note that the cancellation of the collinear terms in the soft limit actually requires the symmetry factor multiplying the collinear term, but not the collinear-soft one. This form of the $\mathbf{A}_1$ operator coincides with that derived in ref. [8] for separating the singly-unresolved kinematical singularities of the squared matrix element at tree-level and is completely universal.

3.2 Counterterms

The expression given in eq. (3.27) is defined only in the strict soft and/or collinear limits. In order to define true countertems, we have to extend it over the whole phase space. This extension requires an exact factorization of the $m+1$ parton phase space into an m parton phase space times the phase space measure of the unresolved parton,

$$\mathrm{d}\phi_{m+1}(\{p\}) = \mathrm{d}\phi_m(\{\tilde{p}\})\,[\mathrm{d}p_1]\,, \tag{3.28}$$

where we introduced the compact notations $\{p\} \equiv \{p_1,\ldots,p_{m+1}\}$ and $\{\tilde{p}\} \equiv \{\tilde{p}_1,\ldots,\tilde{p}_m\}$. Then the subtraction term that regularizes the kinematical singularities of the real-virtual cross section can symbolically be written as

$$\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A}_1} = \mathrm{d}\phi_m[\mathrm{d}p_1]\,\boldsymbol{\mathcal{A}}_1\,2\,\mathrm{Re}\langle\mathcal{M}_{m+1}^{(0)}||\mathcal{M}_{m+1}^{(1)}\rangle\,. \tag{3.29}$$

where we decompose the subtraction term as follows,

$$\boldsymbol{\mathcal{A}}_1\,2\,\mathrm{Re}\langle\mathcal{M}_{m+1}^{(0)}||\mathcal{M}_{m+1}^{(1)}\rangle = \sum_r\left[\sum_{i\neq r}\frac{1}{2}\mathcal{C}_{ir}^{(0,1)}(\{p\}) + \left(\mathcal{S}_r^{(0,1)}(\{p\}) - \sum_{i\neq r}\mathcal{C}_{ir}\mathcal{S}_r^{(0,1)}(\{p\})\right)\right]$$
$$+ \sum_r\left[\sum_{i\neq r}\frac{1}{2}\mathcal{C}_{ir}^{(1,0)}(\{p\}) + \left(\mathcal{S}_r^{(1,0)}(\{p\}) - \sum_{i\neq r}\mathcal{C}_{ir}\mathcal{S}_r^{(1,0)}(\{p\})\right)\right]\,.$$
$$\tag{3.30}$$

All terms above are functions of the original $m + 1$ momenta that enter the one-loop squared matrix element. The last terms in each line on the right hand side do not refer to the collinear limit of anything, but denote functions of the original momenta for which the notation inherits the operator structure of taking the various limits, but otherwise has nothing to do with taking limits.

We now turn to the definition of each term in eq. (3.30). Each term will have the structure that a singular function (Altarelli-Parisi kernel or eikonal factor) is sandwiched between amplitudes. Both the singular functions and the squared matrix elements have their own loop expansions. The double superscipt on the subtraction terms refers to the number of loops in these loop expansions, the first one in the loop expansion of the singular factor while the second one in the expansion of the squared matrix element.

3.2.1 Collinear counterterms

The collinear counterterms are

$$\mathcal{C}_{ir}^{(0,1)}(\{p\}) = 8\pi\alpha_s\mu^{2\varepsilon}\frac{1}{s_{ir}}2\,\mathrm{Re}\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})|\hat{P}_{f_if_r}^{(0)}(z_{i,r},z_{r,i},k_{\perp,i,r};\varepsilon)|\mathcal{M}_m^{(1)}(\{\tilde{p}\}^{(ir)})\rangle\,, \quad (3.31)$$

$$\mathcal{C}_{ir}^{(1,0)}(\{p\}) = (8\pi\alpha_s\mu^{2\varepsilon})^2\frac{1}{s_{ir}^{1+\varepsilon}}c_\Gamma\cos(\pi\varepsilon)\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})|\hat{P}_{f_if_r}^{(1)}(z_{i,r},z_{r,i},k_{\perp,i,r};\varepsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})\rangle. \tag{3.32}$$

The momentum fractions $z_{i,r}$ and $z_{r,i}$ are

$$z_{i,r} = \frac{y_{iQ}}{y_{(ir)Q}} \qquad \text{and} \qquad z_{r,i} = \frac{y_{rQ}}{y_{(ir)Q}}\,, \tag{3.33}$$

while the transverse momentum $k_{\perp,i,r}$ is

$$k_{\perp,i,r}^\mu = \zeta_{i,r}p_r^\mu - \zeta_{r,i}p_i^\mu + \zeta_{ir}\tilde{p}_{ir}^\mu\,, \qquad \zeta_{i,r} = z_{i,r} - \frac{y_{ir}}{\alpha_{ir}y_{(ir)Q}}\,, \quad \zeta_{r,i} = z_{r,i} - \frac{y_{ir}}{\alpha_{ir}y_{(ir)Q}}\,. \tag{3.34}$$

We used the abbreviations $y_{ir} = s_{ir}/Q^2 \equiv 2p_i\cdot p_r/Q^2$, $y_{(ir)Q} = y_{iQ} + y_{rQ}$ with $y_{iQ} = 2p_i\cdot Q/Q^2$, $y_{rQ} = 2p_r\cdot Q/Q^2$ and Q^μ is the total four-momentum of the incoming electron and positron, while $\tilde{p}_{ir}^\mu$ and α_{ir} are defined below in eqs. (3.36) and (3.37) respectively. This choice for the transverse momentum is exactly perpendicular to the parent momentum $\tilde{p}_{ir}^\mu$ and ensures that in the collinear limit $p_i^\mu||p_r^\mu$, the square of $k_{\perp,i,r}^\mu$ behaves as

$$k_{\perp,i,r}^2 \simeq -s_{ir}z_{r,i}z_{i,r}\,, \tag{3.35}$$

as required (independently of ζ_{ir}). In our computation the longitudinal component, proportional to ζ_{ir}, does not contribute due to gauge invariance of the matrix elements, therefore, we may choose $\zeta_{ir} = 0$. The m momenta $\{\tilde{p}\}^{(ir)} \equiv \{\tilde{p}_1,\ldots,\tilde{p}_{ir},\ldots,\tilde{p}_{m+1}\}$ entering the matrix elements on the right hand side of eqs. (3.31) and (3.32) are

$$\tilde{p}_{ir}^\mu = \frac{1}{1-\alpha_{ir}}(p_i^\mu + p_r^\mu - \alpha_{ir}Q^\mu)\,, \qquad \tilde{p}_n^\mu = \frac{1}{1-\alpha_{ir}}p_n^\mu\,, \qquad n \neq i, r\,, \tag{3.36}$$

where

$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y_{(ir)Q}^2 - 4y_{ir}} \right].\tag{3.37}$$

This momentum mapping leads to an exact factorization of the phase space in the form of eq. (3.28). The explicit expression for $[\mathrm{d}p_1]$ reads

$$[\mathrm{d}p_{1;m}^{(ir)}(p_r,\tilde{p}_{ir};Q)] = \mathcal{J}_{1;m}^{(ir)}(p_r,\tilde{p}_{ir};Q)\,\frac{\mathrm{d}^d p_r}{(2\pi)^{d-1}}\delta_+(p_r^2)\,,\tag{3.38}$$

where the Jacobian is

$$\mathcal{J}_{1;m}^{(ir)}(p_r,\tilde{p}_{ir};Q) = y_{\widetilde{ir}\,Q}\,\frac{(1-\alpha_{ir})^{(m-1)(d-2)-1}\,\Theta(1-\alpha_{ir})}{2(1-y_{\widetilde{ir}\,Q})\alpha_{ir} + y_{r\,\widetilde{ir}} + y_{\widetilde{ir}\,Q} - y_{rQ}}\,.\tag{3.39}$$

In this equation α_{ir} has to be expressed as a function of the variable $\tilde{p}_{ir}$,

$$\alpha_{ir} = \frac{\sqrt{(y_{r\,\widetilde{ir}} + y_{\widetilde{ir}\,Q} - y_{rQ})^2 + 4y_{r\,\widetilde{ir}}(1-y_{\widetilde{ir}\,Q})} - (y_{r\,\widetilde{ir}} + y_{\widetilde{ir}\,Q} - y_{rQ})}{2(1-y_{\widetilde{ir}\,Q})}\,.\tag{3.40}$$

It is straightforward to compute the ε expansion of the collinear counterterms that we shall use later. We decompose the expansion into singular terms, finite contributions and terms that vanish as $\varepsilon \to 0$,

$$\mathcal{C}_{ir}^{(0,1)}(\{p\}) = \mathcal{P}oles\,\mathcal{C}_{ir}^{(0,1)}(\{p\}) + \mathcal{F}in\,\mathcal{C}_{ir}^{(0,1)}(\{p\}) + \mathrm{O}(\varepsilon)\,.\tag{3.41}$$

The pole parts can be written in the following unified form:

$$\mathcal{P}oles\,\mathcal{C}_{ir}^{(0,1)}(\{p\}) = -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{s_{ir}}$$
$$\times\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})|\,\boldsymbol{I}(\{\tilde{p}\}^{(ir)};\varepsilon)\,\hat{P}_{f_i f_r}^{(0)}(z_{i,r},z_{r,i},k_{\perp,i,r};\varepsilon)\,|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})\rangle\,,\tag{3.42}$$

where[4]

$$\boldsymbol{I}(\{\tilde{p}\}^{(ir)};\varepsilon) = \frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left(\frac{\mu^2}{Q^2}\right)^\varepsilon\sum_i\left(\boldsymbol{T}_i^2\frac{1}{\varepsilon^2} + \gamma_i\frac{1}{\varepsilon} + \sum_{k\neq i}\boldsymbol{T}_i\boldsymbol{T}_k\frac{1}{\varepsilon}\ln y_{\widetilde{ik}}\right)\tag{3.43}$$

with the usual flavour constants

$$\gamma_q = \frac{3}{2}C_{\mathrm{F}}\,,\qquad \gamma_g = \frac{\beta_0}{2}\,.\tag{3.44}$$

The poles of the $\mathcal{C}_{ir}^{(1,0)}(\{p\})$ counterterm can be written as

$$\mathcal{P}oles\,\mathcal{C}_{ir}^{(1,0)}(\{p\}) = -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{s_{ir}}\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left[(\boldsymbol{T}_i^2+\boldsymbol{T}_r^2-\boldsymbol{T}_{ir}^2)\left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln y_{ir}\right) + \frac{1}{\varepsilon}\left(\gamma_i+\gamma_r-\gamma_{ir}\right)\right.$$
$$\left. -\frac{1}{\varepsilon}\left((\boldsymbol{T}_i^2-\boldsymbol{T}_r^2+\boldsymbol{T}_{ir}^2)\ln z_{i,r} + (\boldsymbol{T}_r^2-\boldsymbol{T}_i^2+\boldsymbol{T}_{ir}^2)\ln z_{r,i}\right)\right]$$
$$\times\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})|\hat{P}_{f_i f_r}^{(0)}(z_{i,r},z_{r,i},k_{\perp,i,r};\varepsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})\rangle\,,\tag{3.45}$$

where $\boldsymbol{T}_{ir} = \boldsymbol{T}_i + \boldsymbol{T}_r$. After the cancellation of the poles is demonstrated (see section 4.2), in a computer code one uses the finite parts of the counterterms. We collect all such finite contributions of the counterterms in appendix A.

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[4]Note that $\boldsymbol{I}(m,\varepsilon) = \boldsymbol{I}(\varepsilon) + \mathrm{O}(\varepsilon^0)$ independently of m.

3.2.2 Soft-type counterterms

We call the soft and soft-collinear counterterms soft-type terms because they all use the momentum mapping appropriate to the soft countertem. We define

$$\mathcal{S}_{r_g}^{(0,1)}(\{p\}) = -8\pi\alpha_s\mu^{2\varepsilon}\sum_i\sum_{k\neq i}\frac{1}{2}\mathcal{S}_{ik}(r)2\,\mathrm{Re}\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|\boldsymbol{T}_i\boldsymbol{T}_k|\mathcal{M}_m^{(1)}(\{\tilde{p}\}^{(r)})\rangle\,, \quad (3.46)$$

$$\mathcal{C}_{ir_g}\mathcal{S}_{r_g}^{(0,1)}(\{p\}) = 8\pi\alpha_s\mu^{2\varepsilon}\frac{1}{s_{ir}}\frac{2z_{i,r}}{z_{r,i}}\boldsymbol{T}_i^2\,2\,\mathrm{Re}\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})||\mathcal{M}_m^{(1)}(\{\tilde{p}\}^{(r)})\rangle\,, \quad (3.47)$$

$$\mathcal{S}_{r_g}^{(1,0)}(\{p\}) = (8\pi\alpha_s\mu^{2\varepsilon})^2c_\Gamma\sum_i\sum_{k\neq i}\frac{1}{2}\mathcal{S}_{ik}(r)$$
$$\times\left[\left(C_A\frac{1}{\varepsilon^2}\frac{\pi\varepsilon}{\sin(\pi\varepsilon)}\left(\frac{1}{2}\mathcal{S}_{ik}(r)\right)^\varepsilon\cos(\pi\varepsilon)+\frac{\beta_0}{2\varepsilon}\frac{\mu^{-2\varepsilon}S_\varepsilon}{(4\pi)^2c_\Gamma}\right)|\mathcal{M}_{m;(i,k)}^{(0)}(\{\tilde{p}\}^{(r)})|^2\right.$$
$$\left.-2\frac{\pi}{\varepsilon}\sum_{l\neq i,k}\left(\frac{1}{2}\mathcal{S}_{kl}(r)\right)^\varepsilon|\mathcal{M}_{m;(i,k,l)}^{(0)}(\{\tilde{p}\}^{(r)})|^2\right]\,, \quad (3.48)$$

$$\mathcal{C}_{ir_g}\mathcal{S}_{r_g}^{(1,0)}(\{p\}) = -(8\pi\alpha_s\mu^{2\varepsilon})^2c_\Gamma\frac{1}{s_{ir}}\frac{2z_{i,r}}{z_{r,i}}\boldsymbol{T}_i^2 \quad (3.49)$$
$$\times\left[\left(C_A\frac{1}{\varepsilon^2}\frac{\pi\varepsilon}{\sin(\pi\varepsilon)}\left(\frac{1}{s_{ir}}\frac{z_{i,r}}{z_{r,i}}\right)^\varepsilon\cos(\pi\varepsilon)+\frac{\beta_0}{2\varepsilon}\frac{\mu^{-2\varepsilon}S_\varepsilon}{(4\pi)^2c_\Gamma}\right)\right]|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|^2\,.$$

The momentum fractions and eikonal functions were defined in eqs. (3.33) and (3.21), while the m momenta $\{\tilde{p}\}^{(r)}\equiv\{\tilde{p}_1,\ldots,\tilde{p}_{m+1}\}$ (p_r is absent) entering the matrix elements on the right hand sides of eqs. (3.46)–(3.49) are defined as

$$\tilde{p}_n^\mu = \Lambda_\nu^\mu[Q,(Q-p_r)/\lambda_r](p_n^\nu/\lambda_r)\,, \qquad n\neq r\,, \quad (3.50)$$

where

$$\lambda_r = \sqrt{1-y_{rQ}}\,, \quad (3.51)$$

and

$$\Lambda_\nu^\mu[K,\widetilde{K}] = g_\nu^\mu - \frac{2(K+\widetilde{K})^\mu(K+\widetilde{K})_\nu}{(K+\widetilde{K})^2} + \frac{2K^\mu\widetilde{K}_\nu}{K^2}\,. \quad (3.52)$$

The matrix $\Lambda_\nu^\mu[K,\widetilde{K}]$ generates a (proper) Lorentz transformation, provided $K^2=\widetilde{K}^2\neq0$. This momenum mapping leads to exact phase-space factorization in the form of eq. (3.28), where

$$[\mathrm{d}p_{1;m}^{(r)}(p_r;Q)] = \mathcal{J}_{1;m}^{(r)}(p_r;Q)\frac{\mathrm{d}^dp_r}{(2\pi)^{d-1}}\delta_+(p_r^2)\,, \quad (3.53)$$

with Jacobian

$$\mathcal{J}_{1;m}^{(r)}(p_r;Q) = \lambda_r^{(m-1)(d-2)-2}\Theta(\lambda_r)\,. \quad (3.54)$$

Finally, we record the pole parts of the soft-type counterterms:

$$\mathcal{P}oles\, \mathcal{S}_{r_g}^{(0,1)}(\{p\}) = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\sum_i\sum_{k\neq i}\frac{1}{4}\mathcal{S}_{ik}(r)$$

$$\times\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|\Big\{\boldsymbol{I}(\{\tilde{p}\}^{(r)};\varepsilon),\boldsymbol{T}_i\boldsymbol{T}_k\Big\}|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})\rangle, \quad (3.55)$$

$$\mathcal{P}oles\, \mathcal{C}_{ir_g}\mathcal{S}_{r_g}^{(0,1)}(\{p\}) = -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{2}{s_{ir}}\frac{z_{i,r}}{z_{r,i}}\boldsymbol{T}_i^2$$

$$\times\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|\,\boldsymbol{I}(\{\tilde{p}\}^{(r)};\varepsilon)\,|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})\rangle, \quad (3.56)$$

$$\mathcal{P}oles\, \mathcal{S}_{r_g}^{(1,0)}(\{p\}) = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\sum_i\sum_{k\neq i}\frac{1}{2}\mathcal{S}_{ik}(r)|\mathcal{M}_{m;(i,k)}^{(0)}(\{\tilde{p}\}^{(r)})|^2$$

$$\times\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left[C_{\mathrm{A}}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{y_{ik}}{y_{ir}y_{kr}}+\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}\right)+\frac{\beta_0}{2\varepsilon}\right], \quad (3.57)$$

$$\mathcal{P}oles\, \mathcal{C}_{ir_g}\mathcal{S}_{r_g}^{(1,0)}(\{p\}) = -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{2}{s_{ir}}\frac{z_{i,r}}{z_{r,i}}\boldsymbol{T}_i^2|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|^2$$

$$\times\frac{\alpha_{\mathrm{s}}}{2\pi}S_\varepsilon\left[C_{\mathrm{A}}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{z_{i,r}}{y_{ir}z_{r,i}}+\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}\right)+\frac{\beta_0}{2\varepsilon}\right]. \quad (3.58)$$

The finite parts are given in appendix A.

4. Counterterms for the integrated approximate cross section

4.1 Factorization in the collinear and soft limits

We wish to construct the approximate cross section $\left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1}$ by the same procedure we used to construct $\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$, therefore, we start by studying the infrared limits of $\langle\mathcal{M}_{m+1}^{(0)}|\boldsymbol{I}(m,\varepsilon)|\mathcal{M}_{m+1}^{(0)}\rangle$ when the momenta of a pair of partons becomes collinear or when a gluon becomes soft.

4.1.1 Collinear limit

In the limit when the momenta of partons i and r become collinear (as defined precisely in eqs. (3.1) and (3.3)) we find[5]

$$\mathbf{C}_{ir}\langle\mathcal{M}_{m+1}^{(0)}(p_i,p_r,\ldots)|\,\boldsymbol{I}(m,\varepsilon)\,|\mathcal{M}_{m+1}^{(0)}(p_i,p_r,\ldots)\rangle = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}$$

$$\times\frac{1}{s_{ir}}\Big(\langle\mathcal{M}_m^{(0)}(p_{ir},\ldots)|\,\boldsymbol{I}(m,\varepsilon)\,\hat{P}_{f_if_r}^{(0)}\,|\mathcal{M}_m^{(0)}(p_{ir},\ldots)\rangle$$

$$+\mathrm{R}_{ir}(y_{ir},z_iy_{(ir)Q},z_ry_{(ir)Q};m,\varepsilon)\langle\mathcal{M}_m^{(0)}(p_{ir},\ldots)|\,\hat{P}_{f_if_r}^{(0)}\,|\mathcal{M}_m^{(0)}(p_{ir},\ldots)\rangle\Big), \quad (4.1)$$

where the function R_{ir} represents those terms that appear in addition to the usual collinear factorization formula of the squared matrix element, due to the presence of the insertion

---

[5]Note that the argument of the insertion operator on the right hand side of eq. (4.1) is the same as the number of coloured external legs in the matrix element.

operator,

$$
\begin{aligned}
\mathrm{R}_{ir}(y_{ir}, z_i y_{(ir)Q}, z_r y_{(ir)Q}; m, \varepsilon) = {} & \frac{\alpha_{\mathrm{s}}}{2\pi} S_\varepsilon \left(\frac{\mu^2}{Q^2}\right)^\varepsilon \\
& \times \Bigg[\mathrm{C}_i(z_i y_{(ir)Q}; m, \varepsilon) \, \boldsymbol{T}_i^2 + \mathrm{C}_r(z_r y_{(ir)Q}; m, \varepsilon) \, \boldsymbol{T}_r^2 - \mathrm{C}_{(ir)}(y_{(ir)Q}; m, \varepsilon) \, \boldsymbol{T}_{ir}^2 \\
& \quad + (\boldsymbol{T}_{ir}^2 - \boldsymbol{T}_i^2 - \boldsymbol{T}_r^2) \, \mathrm{S}_{ir}(y_{ir}, z_i y_{(ir)Q}, z_r y_{(ir)Q}; m, \varepsilon) \Bigg].
\end{aligned}
\tag{4.2}
$$

The m parton matrix elements on the right hand side are obtained from the $m+1$ parton matrix elements by removing partons i and r and replacing them with a single parton denoted by ir in the usual way.

Note that the existence of a universal collinear factorization formula as given in eq. (4.1) is not guaranteed by the factorization properties of QCD matrix elements, but depends also on the particular definition of the subtraction term $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$, which determines the functional dependence of the insertion operator on the momenta. In being able to derive eq. (4.1) it is crucially important that the part of $\boldsymbol{I}(m, \varepsilon)$ that contains true colour-correlations, that is $\mathrm{S}_{il}(y_{il}, y_{iQ}, y_{lQ}; m, \varepsilon)$, depends on its arguments only through the combination

$$
\frac{y_{il}}{y_{iQ} y_{lQ}}.
\tag{4.3}
$$

This expression is independent of the momentum fractions in the collinear limit $p_i \| p_r$. Consequently, the functions S_{il} and S_{rl} have the same limit as p_i and p_r become collinear,

$$
\mathbf{C}_{ir} \mathrm{S}_{il} = \mathbf{C}_{ir} \mathrm{S}_{rl}.
\tag{4.4}
$$

This is important because coherent soft-gluon emission from unresolved partons implies that only the sum of $|\mathcal{M}_{m+1;(i,l)}^{(0)}|^2 + |\mathcal{M}_{m+1;(r,l)}^{(0)}|^2$ (or $|\mathcal{M}_{m+1;(j,i)}^{(0)}|^2 + |\mathcal{M}_{m+1;(j,r)}^{(0)}|^2$) factorizes as

$$
\mathbf{C}_{ir}\left(|\mathcal{M}_{m+1;(i,l)}^{(0)}|^2 + |\mathcal{M}_{m+1;(r,l)}^{(0)}|^2\right) = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon} \frac{1}{s_{ir}} \langle \mathcal{M}_m^{(0)} | \boldsymbol{T}_{ir} \boldsymbol{T}_l \hat{P}_{f_i f_r}^{(0)} | \mathcal{M}_m^{(0)} \rangle.
\tag{4.5}
$$

Then, if and only if eq. (4.4) is fulfilled, we can combine the collinear limits as

$$
\mathbf{C}_{ir}\left(\mathrm{S}_{il}|\mathcal{M}_{m+1;(i,l)}^{(0)}|^2 + \mathrm{S}_{rl}|\mathcal{M}_{m+1;(r,l)}^{(0)}|^2\right) = 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon} \mathrm{S}_{(ir)l} \frac{1}{s_{ir}} \langle \mathcal{M}_m^{(0)} | \boldsymbol{T}_{ir} \boldsymbol{T}_l \hat{P}_{f_i f_r}^{(0)} | \mathcal{M}_m^{(0)} \rangle.
\tag{4.6}
$$

The insertion operators used in other general NLO schemes do not possess this property.

4.1.2 Soft limit

In computing the limit of $\langle \mathcal{M}_{m+1}^{(0)} | \boldsymbol{I}(m, \varepsilon) | \mathcal{M}_{m+1}^{(0)} \rangle$ as the momentum of parton r becomes soft, we need the soft factorization formula for the colour-correlated tree amplitudes as can

be found in ref. [8]. One finds

$$\mathbf{S}_r \langle \mathcal{M}^{(0)}_{m+1}(p_r, \ldots) | \, \boldsymbol{I}(m, \varepsilon) \, | \mathcal{M}^{(0)}_{m+1}(p_r, \ldots) \rangle = -8\pi\alpha_s \mu^{2\varepsilon} \sum_i \sum_{k \neq i} \frac{1}{2} \mathcal{S}_{ik}(r)$$

$$\times \Bigg(\langle \mathcal{M}^{(0)}_m(\ldots) | \frac{1}{2} \{ \boldsymbol{I}(m, \varepsilon), \boldsymbol{T}_i \boldsymbol{T}_k \} | \mathcal{M}^{(0)}_m(\ldots) \rangle$$

$$+ \mathrm{R}_{ik,r}(y_{ik}, y_{ir}, y_{kr}, y_{iQ}, y_{kQ}, y_{rQ}; m, \varepsilon) | \mathcal{M}^{(0)}_{m;(i,k)}(\ldots) |^2 \Bigg) \qquad (4.7)$$

if r is a gluon and $\mathbf{S}_r \langle \mathcal{M}^{(0)}_{m+1} | \boldsymbol{I}(m, \varepsilon) | \mathcal{M}^{(0)}_{m+1} \rangle = 0$ if r is a quark or antiquark. The m parton matrix elements on the right hand side are obtained from the original $m+1$ parton matrix elements by simply removing parton r. In eq. (4.7) the function

$$\mathrm{R}_{ik,r}(y_{ik}, y_{ir}, y_{kr}, y_{iQ}, y_{kQ}, y_{rQ}; m, \varepsilon) = C_A \frac{\alpha_s}{2\pi} S_\varepsilon \left(\frac{\mu^2}{Q^2} \right)^\varepsilon \Big(\mathrm{C}_g(y_{rQ}; m, \varepsilon)$$

$$+ \mathrm{S}_{ik}(y_{ik}, y_{iQ}, y_{kQ}; m, \varepsilon) - \mathrm{S}_{ir}(y_{ir}, y_{iQ}, y_{rQ}; m, \varepsilon) - \mathrm{S}_{rk}(y_{rk}, y_{rQ}, y_{kQ}; m, \varepsilon) \Big) \quad (4.8)$$

represents those terms that appear in addition to the usual soft factorization formula of the squared matrix element due to the presence of the insertion opeartor.

4.1.3 Matching the collinear and soft limits

The collinear limit of the soft factorization formula eq. (4.7) reads

$$\mathbf{C}_{ir} \mathbf{S}_r \langle \mathcal{M}^{(0)}_{m+1}(p_i, p_r, \ldots) | \, \boldsymbol{I}(m, \varepsilon) \, | \mathcal{M}^{(0)}_{m+1}(p_i, p_r, \ldots) \rangle = 8\pi\alpha_s \mu^{2\varepsilon} \frac{2}{s_{ir}} \frac{z_i}{z_r} \boldsymbol{T}_i^2$$

$$\times \Big(\langle \mathcal{M}^{(0)}_m(p_i, \ldots) | \, \boldsymbol{I}(m, \varepsilon) \, | \mathcal{M}^{(0)}_m(p_i, \ldots) \rangle + | \mathcal{M}^{(0)}_m(p_i, \ldots) |^2 \mathbf{C}_{ir} \mathrm{R}_{ik,r} \Big), \quad (4.9)$$

with

$$\mathbf{C}_{ir} \mathrm{R}_{ik,r}(y_{ik}, y_{ir}, y_{kr}, y_{iQ}, y_{kQ}, y_{rQ}; m, \varepsilon) = C_A \frac{\alpha_s}{2\pi} S_\varepsilon \left(\frac{\mu^2}{Q^2} \right)^\varepsilon$$

$$\times \Big(\mathrm{C}_g(z_r y_{(ir)Q}; m, \varepsilon) - \mathrm{S}_{ir}(y_{ir}, z_i y_{(ir)Q}, z_r y_{(ir)Q}; m, \varepsilon) \Big). \qquad (4.10)$$

The soft limit of the collinear factorization formula eq. (4.1) is

$$\mathbf{S}_r \mathbf{C}_{ir} \langle \mathcal{M}^{(0)}_{m+1}(p_i, p_r, \ldots) | \, \boldsymbol{I}(m, \varepsilon) \, | \mathcal{M}^{(0)}_{m+1}(p_i, p_r, \ldots) \rangle = 8\pi\alpha_s \mu^{2\varepsilon} \frac{2}{s_{ir}} \frac{1}{z_r} \boldsymbol{T}_i^2$$

$$\times \Big(\langle \mathcal{M}^{(0)}_m(p_i, \ldots) | \, \boldsymbol{I}(m, \varepsilon) \, | \mathcal{M}^{(0)}_m(p_i, \ldots) \rangle + | \mathcal{M}^{(0)}_m(p_i, \ldots) |^2 \mathbf{S}_r \mathrm{R}_{ir} \Big). \quad (4.11)$$

with

$$\mathbf{S}_r \mathrm{R}_{ir}(y_{ir}, z_i y_{(ir)Q}, z_r y_{(ir)Q}; m, \varepsilon) = C_A \frac{\alpha_s}{2\pi} S_\varepsilon \left(\frac{\mu^2}{Q^2} \right)^\varepsilon$$

$$\times \Big(\mathrm{C}_g(z_r y_{(ir)Q}; m, \varepsilon) - \mathrm{S}_{ir}(y_{ir}, y_{(ir)Q}, z_r y_{(ir)Q}; m, \varepsilon) \Big). \qquad (4.12)$$

– 18 –

Thus the same arguments as below eq. (3.24) apply here as well, therefore,

$$\mathbf{C}_{ir}(\mathbf{S}_r - \mathbf{C}_{ir}\mathbf{S}_r)\langle\mathcal{M}^{(0)}_{m+1}|\boldsymbol{I}(m,\varepsilon)|\mathcal{M}^{(0)}_{m+1}\rangle = 0\,, \tag{4.13}$$

$$\mathbf{S}_r(\mathbf{C}_{ir} - \mathbf{C}_{ir}\mathbf{S}_r)\langle\mathcal{M}^{(0)}_{m+1}|\boldsymbol{I}(m,\varepsilon)|\mathcal{M}^{(0)}_{m+1}\rangle = 0 \tag{4.14}$$

and our candidate counterterm has the same structure as in eq. (3.27),

$$\mathbf{A}_1\,\langle\mathcal{M}^{(0)}_{m+1}|\boldsymbol{I}(m,\varepsilon)|\mathcal{M}^{(0)}_{m+1}\rangle =$$
$$= \sum_r\left[\sum_{i\neq r}\frac{1}{2}\mathbf{C}_{ir} + \left(\mathbf{S}_r - \sum_{i\neq r}\mathbf{C}_{ir}\mathbf{S}_r\right)\right]\left(|\mathcal{M}^{(0)}_{m+1}(p_i,p_r,\ldots)|^2\otimes\boldsymbol{I}(m,\varepsilon)\right)\,. \tag{4.15}$$

As before the cancellation of the collinear terms in the soft limit requires the symmetry factor multiplying the collinear term, but not the collinear-soft one.

4.2 Counterterms

In order to define the counterterms, we extend eq. (4.15) over the whole phase space as done in section 3.2. We introduce the phase space factorization as in eq. (3.28) and write the subtraction term that regularizes the kinematical singularities of the integrated approximate cross section in the symbolic form

$$\left(\int_1\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}\right)^{\mathrm{A}_1} = \mathrm{d}\phi_m[\mathrm{d}p_1]\boldsymbol{\mathcal{A}}_1\left(|\mathcal{M}^{(0)}_{m+1}|^2\otimes\boldsymbol{I}(m,\varepsilon)\right)\,, \tag{4.16}$$

where

$$\boldsymbol{\mathcal{A}}_1\left(|\mathcal{M}^{(0)}_{m+1}|^2\otimes\boldsymbol{I}(m,\varepsilon)\right) =$$
$$= \sum_r\left[\sum_{i\neq r}\frac{1}{2}\mathcal{C}^{(0,0\otimes I)}_{ir}(\{p\}) + \left(\mathcal{S}^{(0,0\otimes I)}_r(\{p\}) - \sum_{i\neq r}\mathcal{C}_{ir}\mathcal{S}^{(0,0\otimes I)}_r(\{p\})\right)\right]$$
$$+ \sum_r\left[\sum_{i\neq r}\frac{1}{2}\mathcal{C}^{R\times(0,0)}_{ir}(\{p\}) + \left(\mathcal{S}^{R\times(0,0)}_r(\{p\}) - \sum_{i\neq r}\mathcal{C}_{ir}\mathcal{S}^{R\times(0,0)}_r(\{p\})\right)\right]\,. \tag{4.17}$$

We now define all terms on the right hand side of this equation precisely. The structure of eq. (4.17) is the same as that of eq. (3.30). Thus defining true subtraction terms starting from the limiting formulae of the previous subsection follows the steps of section 3.2.

4.2.1 Collinear counterterms

The collinear subtraction terms read

$$\mathcal{C}^{(0,0\otimes I)}_{ir}(\{p\}) = 8\pi\alpha_\mathrm{s}\mu^{2\varepsilon}\frac{1}{s_{ir}} \tag{4.18}$$
$$\times\langle\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir)})|\,\boldsymbol{I}(\{\tilde{p}\}^{(ir)};m,\varepsilon)\,\hat{P}^{(0)}_{f_if_r}(z_{i,r},z_{r,i},k_{\perp,i,r};\varepsilon)\,|\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir)})\rangle$$

and

$$\mathcal{C}^{R\times(0,0)}_{ir}(\{p\}) = 8\pi\alpha_\mathrm{s}\mu^{2\varepsilon}\frac{1}{s_{ir}}\mathrm{R}_{ir}(y_{ir},z_{i,r}y_{\widetilde{ir}\,Q},z_{r,i}y_{\widetilde{ir}\,Q};m,\varepsilon) \tag{4.19}$$
$$\times\langle\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir)})|\hat{P}^{(0)}_{f_if_r}(z_{i,r},z_{r,i},k_{\perp,i,r};\varepsilon)|\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir)})\rangle\,.$$

– 19 –

Computing the pole parts of eqs. (4.18) and (4.19), we can easily see that

$$\mathcal{P}oles\left[\mathcal{C}_{ir}^{(0,0\otimes I)}(\{p\}) + \mathcal{C}_{ir}^{(0,1)}(\{p\})\right] = 0 \tag{4.20}$$

and

$$\mathcal{P}oles\left[\mathcal{C}_{ir}^{R\times(0,0)}(\{p\}) + \mathcal{C}_{ir}^{(1,0)}(\{p\})\right] = 0. \tag{4.21}$$

Therefore, the sum of eqs. (3.31) and (4.18) as well as that of eqs. (3.32) and (4.19) is finite in $d = 4$ dimensions.

4.2.2 Soft-type counterterms

The soft-type counterterms are defined as

$$\begin{aligned}
\mathcal{S}_{r_g}^{(0,0\otimes I)}(\{p\}) = {}& -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\sum_i\sum_{k\neq i}\frac{1}{4}\mathcal{S}_{ik}(r) \\
&\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|\{\boldsymbol{I}(\{\tilde{p}\}^{(r)};m,\varepsilon),\boldsymbol{T}_i\boldsymbol{T}_k\}|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})\rangle,
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\mathcal{C}_{ir_g}\mathcal{S}_{r_g}^{(0,0\otimes I)}(\{p\}) = {}& 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{s_{ir}}\frac{2z_{i,r}}{z_{r,i}} \\
&\times \boldsymbol{T}_i^2\,\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|\boldsymbol{I}(\{\tilde{p}\}^{(r)};m,\varepsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})\rangle,
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
\mathcal{S}_{r_g}^{R\times(0,0)}(\{p\}) = {}& -8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\sum_i\sum_{k\neq i}\frac{1}{2}\mathcal{S}_{ik}(r)\,|\mathcal{M}_{m;(i,k)}^{(0)}(\{\tilde{p}\}^{(r)})|^2 \\
&\times \mathrm{R}_{ik,r}(y_{ik},y_{ir},y_{kr},y_{iQ},y_{kQ},y_{rQ};m,\varepsilon),
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\mathcal{C}_{ir_g}\mathcal{S}_{r_g}^{R\times(0,0)}(\{p\}) = {}& 8\pi\alpha_{\mathrm{s}}\mu^{2\varepsilon}\frac{1}{s_{ir}}\frac{2z_{i,r}}{z_{r,i}}\boldsymbol{T}_i^2|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|^2 C_{\mathrm{A}}\frac{\alpha_{\mathrm{s}}}{2\pi}\,S_\varepsilon\left(\frac{\mu^2}{Q^2}\right)^\varepsilon \\
&\times \Big(\mathrm{C}_g(z_{r,i}y_{(ir)Q};m,\varepsilon) - \mathrm{S}_{ir}(y_{ir},z_{i,r}y_{(ir)Q},z_{r,i}y_{(ir)Q};m,\varepsilon)\Big).
\end{aligned} \tag{4.25}$$

We can simplify the arguments in the second line of eq. (4.25) using eq. (3.33),

$$\mathrm{C}_g(z_{r,i}y_{(ir)Q};m,\varepsilon) - \mathrm{S}_{ir}(y_{ir},z_{i,r}y_{(ir)Q},z_{r,i}y_{(ir)Q};m,\varepsilon) = \mathrm{C}_g(y_{rQ};m,\varepsilon) - \mathrm{S}_{ir}(y_{ir},y_{iQ},y_{rQ};m,\varepsilon). \tag{4.26}$$

Similarly to the collinear cases, the pole parts cancel term by term between eqs. (3.46)–(3.49) and eqs. (4.22)–(4.25),
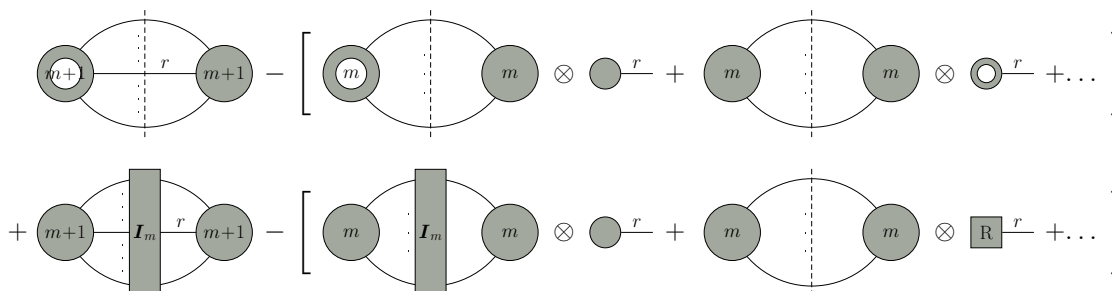
$$\mathcal{P}oles\left[\mathcal{S}_r^{(0,0\otimes I)}(\{p\}) + \mathcal{S}_r^{(0,1)}(\{p\})\right] = 0, \tag{4.27}$$

$$\mathcal{P}oles\left[\mathcal{C}_{ir}\mathcal{S}_r^{(0,0\otimes I)}(\{p\}) + \mathcal{C}_{ir}\mathcal{S}_r^{(0,1)}(\{p\})\right] = 0, \tag{4.28}$$

$$\mathcal{P}oles\left[\mathcal{S}_r^{R\times(0,0)}(\{p\}) + \mathcal{S}_r^{(1,0)}(\{p\})\right] = 0, \tag{4.29}$$

$$\mathcal{P}oles\left[\mathcal{C}_{ir}\mathcal{S}_r^{R\times(0,0)}(\{p\}) + \mathcal{C}_{ir}\mathcal{S}_r^{(1,0)}(\{p\})\right] = 0. \tag{4.30}$$

Consequently, the sum of eqs. (3.46) and (4.22), that of eqs. (3.47) and (4.23), that of eqs. (3.48) and (4.24) and that of eqs. (3.49) and (4.25) are finite in $d = 4$ dimensions.

Figure 1: Graphical representation of the squared matrix element and the subtraction terms.

We help the reader in grasping the various cancellations by visualizing the subtraction terms in figure 1. The first picture corresponds to the squared matrix element of the $m+1$ final-state partons. The fully shaded circle represents the Born amplitude, while the circle with a whole is the one-loop amplitude. The following terms in the first squared brackets correspond to the terms that build $\boldsymbol{\mathcal{A}}_1 \, 2\operatorname{Re}\langle\mathcal{M}^{(0)}_{m+1}||\mathcal{M}^{(1)}_{m+1}\rangle$, defined in eq. (3.30); the first one represents (0,1)-type terms like $\mathcal{C}^{(0,1)}_{ir}(\{p\})$ and the second is for (1,0)-type terms such as $\mathcal{C}^{(1,0)}_{ir}(\{p\})$. The first picture in the second line is the result of the integration in eq. (2.21). Finally, the terms in the second squared brackets represent terms that contribute to $\boldsymbol{\mathcal{A}}_1\Big(|\mathcal{M}^{(0)}_{m+1}|^2 \otimes \boldsymbol{I}(m,\varepsilon)\Big)$, defined in eq. (4.17); the first representing terms of the $(0,0\otimes\boldsymbol{I})$-type, such as $\mathcal{C}^{(0,0\otimes I)}_{ir}(\{p\})$, while the second one stands for $R\times(0,0)$-type terms like $\mathcal{C}^{R\times(0,0)}_{ir}(\{p\})$. The factorized one-parton factors correspond to Altarelli-Parisi kernels, with azimuthal correlations included, or eikonal factors, with colour-correlations included, either at tree-level (fully shaded circles), or at one-loop (circles with holes), or R-functions (boxes).

The cancellations of the ε poles occurs vertically, term by term as shown previously. The regularization of the kinematical singularities takes place horizontally, between the first term and the terms in the following brackets, separately in each line. Kinematical singularities, introduced by the subtractions terms, are screened by the jet function, J_m, just as in NLO computations.

We conclude that the $(m + 1)$-parton contribution in eq. (1.4) is free of ε poles as well as unscreened kinematical singularities. We can set $\varepsilon = 0$ and compute the integral of $\mathrm{d}\sigma^{\mathrm{NNLO}}_{m+1}$ by standard Monte Carlo methods. Such an integration uses the finite parts in the ε-expansion of the subtraction terms as $\varepsilon \to 0$. The finite parts of the integrated approximate cross section of eq. (2.21) can be found in ref. [11],[6] while those of eq. (3.30) together with the finite parts of the counterterms in eq. (4.17) are presented in appendix A.

5. Numerical results

In this paper we have defined explicitly all subtraction terms that are needed to make $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$ integrable in $d = 4$ dimensions for processes without coloured partons in the initial state.

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[6]Notice the shift of m by one unit in this paper ($m \to m + 1$) as compared to the value in ref. [11].

n	$C_{\tau;4}^{(n)}$	$C_{C;4}^{(n)}$
1	-$(1.23 \pm 0.01) \cdot 10^3$	-$(4.33 \pm 0.05) \cdot 10^3$
2	-$(2.55 \pm 0.01) \cdot 10^2$	-$(3.25 \pm 0.02) \cdot 10^3$
3	-$(4.79 \pm 0.03) \cdot 10^1$	-$(1.80 \pm 0.01) \cdot 10^3$

Table 1: The moments $C_{\tau;4}^{(n)}$ and $C_{C;4}^{(n)}$.

We have proven the cancellation of the ε poles explicitly. In order to demonstrate that the subtraction terms indeed regularize the cross section for the real-virtual correction, we consider the contribution to the theoretical predictions for the three-jet event-shape distributions thrust (T) and C-parameter in electron-positron annihilation, when the jet function is a functional

$$J_n(p_1, \ldots, p_n; O) = \delta(O - O_3(p_1, \ldots, p_n)), \quad (5.1)$$

with $O_3(p_1, \ldots, p_n)$ being the value of either $\tau \equiv 1 - T$ or C for a given event $(p_1, \ldots, p_n)$. Starting from randomly chosen phase space points and approaching the singly-unresolved (soft and/or collinear) regions of the phase space in successive steps, we have checked numerically that the sum of the subtraction terms has the same limits (up to integrable square-root singularities) as the squared matrix element itself.

The perturbative expansion of the nth moment of a three-jet observable at a fixed scale $\mu = Q$ and NNLO accuracy can be parametrised as

$$\langle O^n \rangle \equiv \int dO\, O^n\, \frac{1}{\sigma_0} \frac{d\sigma}{dO}(O) = \left(\frac{\alpha_s(Q)}{2\pi}\right) A_O^{(n)} + \left(\frac{\alpha_s(Q)}{2\pi}\right)^2 B_O^{(n)} + \left(\frac{\alpha_s(Q)}{2\pi}\right)^3 C_O^{(n)}, \quad (5.2)$$

where according to eq. (1.2), the NNLO correction is a sum of three contributions,

$$C_O^{(n)} = C_{O;5}^{(n)} + C_{O;4}^{(n)} + C_{O;3}^{(n)}. \quad (5.3)$$

Carrying out the phase space integrations in eq. (1.4), we computed the four-parton contribution $C_{O;4}^{(n)}$ as defined in this article. The predictions for the first three moments of τ and the C-parameter, obtained using about four million Monte Carlo events, are presented in table 1. In performing the numerical integrations, we do not encounter more severe numerical problems than known from computing the real-emission contribution in NLO computations and the computation of differential distributions does not pose any problem. The required CPU time is however much longer because of the much more cumbersome expressions that enter the various loop matrix elements.

6. Conclusions

In a companion paper [10] we generalized the dipole subtraction scheme to computing NNLO corrections to QCD jet cross sections to processes without coloured partons in the initial state. The scheme is completely general in the sense that any number of massless coloured final-state partons (massive vector bosons are assumed to decay into massless fermions) are allowed provided the necessary squared matrix elements are known.

Three types of corrections contribute to the NNLO corrections: the doubly-real, the real-virtual and the doubly-virtual ones. In ref. [10] we defined the subtraction terms for the doubly-real emissions; those to the real-virtual correction can be found in the present paper. By rendering these two contributions finite in $d = 4$ dimensions, the KLN theorem ensures that for infrared safe observables adding the subtractions above to the doubly-virtual correction that becomes also finite in $d = 4$ dimensions.

The subtraction terms for the real-virtual corrections presented here are local in $d = 4-2\varepsilon$ dimensions, include complete colour and azimuthal correlations. The expressions were derived by extending the singly-unresolved limits of the one-loop squared matrix elements over the whole phase space and also extending the singly-unresolved limits of the integrated approximate cross section, used for regularizing the kinematical singularities of the cross section for doubly-real emmissions over the singly-unresolved regions of the phase space.

In order to demonstrate that the subtracted cross section is indeed integrable, we have computed the corresponding contributions to the first three moments of two three-jet event-shape observables, the thrust and the C-parameter. In performing the numerical integrations, we do not encounter more severe numerical problems than known from NLO computations. The required CPU time is however much longer because of the much more cumbersome expressions that enter the various loop matrix elements.

Acknowledgments

A. Finite parts of the subtraction terms

In this appendix we present the finite parts, as defined in the decomposition (3.41), of eqs. (3.30) and (4.17), term by term. We start with the collinear counterterms. The finite part of the sum of eqs. (3.31) and (4.18) is

$$
\mathcal{F}in\left[\mathcal{C}_{ir}^{(0,1)}(\{p\}) + \mathcal{C}_{ir}^{(0,0\otimes I)}(\{p\})\right] = 4\alpha_{\mathrm{s}}^2 \frac{1}{s_{ir}}
$$

$$
\times \Bigg[2\,\mathrm{Re}\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})|\,\hat{P}_{f_i f_r}^{(0)}(z_{i,r}, z_{r,i}, k_{\perp,i,r})\,\mathcal{F}in\,|\mathcal{M}_m^{(1)}(\{\tilde{p}\}^{(ir)})\rangle
$$

$$
+ \sum_i \Bigg(\sum_{k\neq i} \mathcal{F}in\left[\mathrm{S}_{ik}(y_{\tilde{i}\tilde{k}}, y_{\tilde{i}Q}, y_{\tilde{k}Q}; m) + \mathrm{CS}(m)\right]
$$

$$
\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})|\,\boldsymbol{T}_i\boldsymbol{T}_k\,\hat{P}_{f_i f_r}^{(0)}(z_{i,r}, z_{r,i}, k_{\perp,i,r})\,|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})\rangle
$$

$$
+ \mathcal{F}in\left[\mathrm{C}_i(y_{\tilde{i}Q}; m) + \mathrm{CS}(m)\right]
$$

$$
\times \boldsymbol{T}_i^2\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})|\,\hat{P}_{f_i f_r}^{(0)}(z_{i,r}, z_{r,i}, k_{\perp,i,r})\,|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})\rangle \Bigg)\Bigg], \quad \text{(A.1)}
$$

where the finite part of the one-loop amplitude is defined by the following equation:

$$|\mathcal{M}_m^{(1)}(\{p\})\rangle = -\frac{1}{2}\boldsymbol{I}(\{p\};\varepsilon)|\mathcal{M}_m^{(0)}(\{p\})\rangle + \frac{\alpha_{\mathrm{s}}}{2\pi}\mathcal{F}in\,|\mathcal{M}_m^{(1)}(\{p\})\rangle\,, \tag{A.2}$$

with $\boldsymbol{I}(\{p\};\varepsilon)$ given in eq. (3.43). The finite parts of the functions $\mathrm{S}_{ik}+\mathrm{CS}$ and $\mathrm{C}_i+\mathrm{CS}$ can be found in the appendix of ref. [11]. Next we turn to the finite parts of eqs. (3.32) and (4.19), which are given separately for the various flavour combinations. For $f_i = f_r = g$:

$$\mathcal{F}in\left[\mathcal{C}_{g_i g_r}^{(1,0)}(\{p\}) + \mathcal{C}_{g_i g_r}^{R\times(0,0)}(\{p\})\right] = C_{\mathrm{A}}\,4\alpha_{\mathrm{s}}^2\,\frac{1}{s_{ir}}$$

$$\times\Bigg[\left(\mathcal{F}in\left[\frac{1}{2}\left(\mathrm{C}_{gg}(z_{i,r}y_{\widetilde{ir}\,Q};m) + \mathrm{C}_{gg}(z_{r,i}y_{\widetilde{ir}\,Q};m) - \mathrm{C}_{gg}(y_{\widetilde{ir}\,Q};m)\right)\right.\right.$$

$$+n_{\mathrm{f}}\left(\mathrm{C}_{q\bar{q}}(z_{i,r}y_{\widetilde{ir}\,Q};m) + \mathrm{C}_{q\bar{q}}(z_{r,i}y_{\widetilde{ir}\,Q};m) - \mathrm{C}_{q\bar{q}}(y_{\widetilde{ir}\,Q};m)\right)$$

$$-\mathrm{S}_{ir}(y_{ir}, z_{i,r}y_{\widetilde{ir}\,Q}, z_{r,i}y_{\widetilde{ir}\,Q};m) - \mathrm{CS}(m)\Bigg]$$

$$\left.-\frac{1}{2}\ln^2\frac{z_{i,r}}{z_{r,i}} - \frac{1}{2}\ln^2 y_{ir} - \ln y_{ir}\ln(z_{i,r}z_{r,i}) + \frac{\pi^2}{3} + \frac{\beta_0}{2C_{\mathrm{A}}}\ln\frac{\mu^2}{Q^2}\right)$$

$$\times\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})|\,\hat{P}_{g_i g_r}^{(0)}(z_{i,r}, z_{r,i}, k_{\perp,i,r})\,|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})\rangle$$

$$-\frac{2}{3}(C_{\mathrm{A}} - 2T_{\mathrm{R}}n_{\mathrm{f}})\,\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})|\,\frac{k_{\perp,i,r}^{\mu}k_{\perp,i,r}^{\nu}}{k_{\perp,i,r}^2}\,|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})\rangle\Bigg]\,. \tag{A.3}$$

For $f_i = q$, $f_r = \bar{q}$:

$$\mathcal{F}in\left[\mathcal{C}_{q_i\bar{q}_r}^{(1,0)}(\{p\}) + \mathcal{C}_{q_i\bar{q}_r}^{R\times(0,0)}(\{p\})\right] = 4\alpha_{\mathrm{s}}^2\,\frac{1}{s_{ir}}$$

$$\times\Bigg(C_{\mathrm{F}}\,\mathcal{F}in\left[\mathrm{C}_{qg}(z_{i,r}y_{\widetilde{ir}\,Q};m) + \mathrm{C}_{qg}(z_{r,i}y_{\widetilde{ir}\,Q};m)\right]$$

$$-C_{\mathrm{A}}\,\mathcal{F}in\left[\frac{1}{2}\mathrm{C}_{gg}(y_{\widetilde{ir}\,Q};m) + n_{\mathrm{f}}\mathrm{C}_{q\bar{q}}(y_{\widetilde{ir}\,Q};m)\right]$$

$$+(C_{\mathrm{A}} - 2C_{\mathrm{F}})\,\mathcal{F}in\left[\mathrm{S}_{ir}(y_{ir}, z_{i,r}y_{\widetilde{ir}\,Q}, z_{r,i}y_{\widetilde{ir}\,Q};m) + \mathrm{CS}(m)\right]$$

$$+(C_{\mathrm{A}} - 2C_{\mathrm{F}})\left(\frac{1}{2}\ln^2 y_{ir} - \frac{\pi^2}{2} + 1\right) + 2(\gamma_g - \gamma_q)(2 - \ln y_{ir})$$

$$-C_{\mathrm{A}}\left(\frac{1}{2}\ln^2\frac{z_{i,r}}{z_{r,i}} + \ln y_{ir}\,\ln(z_{i,r}\,z_{r,i}) + \frac{\pi^2}{6}\right) + \frac{1}{9}(C_{\mathrm{A}} + 4T_{\mathrm{R}}n_{\mathrm{f}})$$

$$+\frac{\beta_0}{2}\ln\frac{\mu^2}{Q^2}\Bigg)\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})|\,\hat{P}_{q_i\bar{q}_r}^{(0)}(z_{i,r}, z_{r,i}, k_{\perp,i,r})\,|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir)})\rangle\,. \tag{A.4}$$

For $f_i = q$, $f_r = g$:

$$\mathcal{F}in\left[\mathcal{C}^{(1,0)}_{q_i g_r}(\{p\}) + \mathcal{C}^{R\times(0,0)}_{q_i g_r}(\{p\})\right] = 4\alpha_s^2 \frac{1}{s_{ir}}$$

$$\times\Bigg[\Bigg(C_F\,\mathcal{F}in\left[\mathrm{C}_{qg}(z_{i,r}y_{\widetilde{ir}Q};m) - \mathrm{C}_{qg}(y_{\widetilde{ir}Q};m)\right]$$

$$+C_A\,\mathcal{F}in\left[\frac{1}{2}\mathrm{C}_{gg}(z_{r,i}y_{\widetilde{ir}Q};m) + n_f\mathrm{C}_{q\bar{q}}(z_{r,i}y_{\widetilde{ir}Q};m)\right]$$

$$-C_A\,\mathcal{F}in\left[\mathrm{S}_{ir}(y_{ir}, z_{i,r}y_{\widetilde{ir}Q}, z_{r,i}y_{\widetilde{ir}Q};m) + \mathrm{CS}(m)\right]$$

$$+2C_F\left(\mathrm{Li}_2\left(-\frac{z_{r,i}}{z_{i,r}}\right) - \ln y_{ir}\,\ln z_{i,r}\right)$$

$$-C_A\left(\frac{1}{2}\ln^2\frac{z_{i,r}}{z_{r,i}} + \frac{1}{2}\ln^2 y_{ir} + \ln y_{ir}\ln\frac{z_{r,i}}{z_{i,r}} + 2\,\mathrm{Li}_2\left(-\frac{z_{r,i}}{z_{i,r}}\right) - \frac{\pi^2}{3}\right)$$

$$+\frac{\beta_0}{2}\ln\frac{\mu^2}{Q^2}\Bigg)\langle\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir)})|\,\hat{P}^{(0)}_{q_i g_r}(z_{i,r}, z_{r,i})\,|\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir)})\rangle$$

$$+C_F(C_A - C_F)|\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir)})|^2\Bigg]. \tag{A.5}$$

The case of $f_i = g$, $f_r = q$ is obtained from the latter with the $i \leftrightarrow r$ substitution.

The finite parts of the soft subtractions are as follows:

$$\mathcal{F}in\left[\mathcal{S}^{(0,1)}_{ir}(\{p\}) + \mathcal{S}^{(0,0\otimes I)}_{ir}(\{p\})\right] = -4\alpha_s^2\sum_i\sum_{k\neq i}\frac{1}{2}\mathcal{S}_{ik}(r)$$

$$\times\Bigg[2\,\mathrm{Re}\langle\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(r)})|\,\boldsymbol{T}_i\boldsymbol{T}_k\,\mathcal{F}in\,|\mathcal{M}^{(1)}_m(\{\tilde{p}\}^{(r)})\rangle$$

$$+\sum_j\left(\sum_{l\neq j}\frac{1}{2}\mathcal{F}in\left[\mathrm{S}_{jl}(y_{\tilde{j}\tilde{l}}, y_{\tilde{j}Q}, y_{\tilde{l}Q};m) + \mathrm{CS}(m)\right]|\mathcal{M}^{(0)}_{m;(i,k)(j,l)}(\{\tilde{p}\}^{(r)})|^2\right.$$

$$\left.+\mathcal{F}in\left[\mathrm{C}_j(y_{\tilde{j}Q};m) + \mathrm{CS}(m)\right]\boldsymbol{T}_j^2\,|\mathcal{M}^{(0)}_{m;(i,k)}(\{\tilde{p}\}^{(r)})|^2\right)\Bigg], \tag{A.6}$$

$$\mathcal{F}in\left[\mathcal{S}^{(1,0)}_{ir}(\{p\}) + \mathcal{S}^{R\times(0,0)}_{ir}(\{p\})\right] = 4\alpha_s^2\sum_i\sum_{k\neq i}\Bigg\{C_A\frac{1}{2}\mathcal{S}_{ik}(r)\,|\mathcal{M}^{(0)}_{m;(i,k)}(\{\tilde{p}\}^{(r)})|^2$$

$$\times\left(\frac{1}{2}\ln^2\frac{y_{ik}}{y_{ir}y_{kr}} - \frac{\pi^2}{3} - \frac{\beta_0}{2C_A}\ln\frac{\mu^2}{Q^2} - \mathcal{F}in\left[\frac{1}{2}\mathrm{C}_{gg}(y_{rQ};m) + n_f\mathrm{C}_{q\bar{q}}(y_{rQ};m)\right]\right.$$

$$\left.-\mathcal{F}in\left[\mathrm{S}_{ik}(y_{ik}, y_{iQ}, y_{kQ};m) - \mathrm{S}_{ir}(y_{ir}, y_{iQ}, y_{rQ};m) - \mathrm{S}_{kr}(y_{kr}, y_{kQ}, y_{rQ};m) - \mathrm{CS}(m)\right]\right)$$

$$-2\pi\sum_{l\neq i,k}\ln\frac{y_{kl}}{y_{kr}y_{lr}}|\mathcal{M}^{(0)}_{m;(i,k,l)}(\{\tilde{p}\}^{(r)})|^2\Bigg\}. \tag{A.7}$$

Finally, we present the finite parts of the collinear-soft subtractions:

$$
\mathcal{F}\!in\left[\mathcal{C}_{ir}\mathcal{S}_r^{(0,1)}(\{p\}) + \mathcal{C}_{ir}\mathcal{S}_r^{(0,0\otimes I)}(\{p\})\right] = 4\alpha_s^2 \frac{2}{s_{ir}}\frac{z_{i,r}}{z_{r,i}} \boldsymbol{T}_i^2
$$

$$
\times\Bigg[2\operatorname{Re}\langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|\,\mathcal{F}\!in\,|\mathcal{M}_m^{(1)}(\{\tilde{p}\}^{(r)})\rangle
$$

$$
+\sum_j\Bigg(\sum_{l\neq j}\mathcal{F}\!in\left[\mathrm{S}_{jl}(y_{\tilde{j}\tilde{l}},y_{\tilde{j}Q},y_{\tilde{l}Q};m)+\mathrm{CS}(m)\right]|\mathcal{M}_{m;(j,l)}^{(0)}(\{\tilde{p}\}^{(r)})|^2
$$

$$
+\mathcal{F}\!in\left[\mathrm{C}_j(y_{\tilde{j}Q};m)+\mathrm{CS}(m)\right]\boldsymbol{T}_j^2\,|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|^2\Bigg), \tag{A.8}
$$

$$
\mathcal{F}\!in\left[\mathcal{C}_{ir}\mathcal{S}_r^{(1,0)}(\{p\}) + \mathcal{C}_{ir}\mathcal{S}_r^{R\times(0,0)}(\{p\})\right] = -4\alpha_s^2\, C_A \frac{2}{s_{ir}}\frac{z_{i,r}}{z_{r,i}} \boldsymbol{T}_i^2|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(r)})|^2
$$

$$
\times\Bigg(\frac{1}{2}\ln^2\frac{z_{i,r}}{y_{ir}z_{r,i}} - \frac{\pi^2}{3} - \frac{\beta_0}{2C_A}\ln\frac{\mu^2}{Q^2} - \mathcal{F}\!in\left[\frac{1}{2}\mathrm{C}_{gg}(y_{rQ};m)+n_f\mathrm{C}_{q\bar{q}}(y_{rQ};m)\right]
$$

$$
+\mathcal{F}\!in\left[\mathrm{S}_{ir}(y_{ir},y_{iQ},y_{rQ};m)+\mathrm{CS}(m)\right]\Bigg). \tag{A.9}
$$

For the sake of completeness, we recall the finite part of the first two terms in eq. (1.4) from ref. [11] adapted to the present case:

$$
\left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} + \mathrm{d}\sigma_{m+1}^{\mathrm{R}}\otimes\boldsymbol{I}(m,\varepsilon)\right]_{\varepsilon=0} = \mathcal{N}\frac{\alpha_s}{2\pi}\sum_{\{m+1\}}\mathrm{d}\phi_{m+1}\frac{1}{S_{\{m+1\}}}
$$

$$
\times\Bigg\{ 2\operatorname{Re}\langle\mathcal{M}_{m+1}^{(0)}(\{p\})|\mathcal{F}\!in\,|\mathcal{M}_{m+1}^{(1)}(\{p\})\rangle
$$

$$
+\sum_i\Bigg[\sum_{k\neq i}\mathcal{F}\!in\,[\mathrm{S}_{ik}(y_{ik},y_{iQ},y_{kQ};m)+\mathrm{CS}(m)]\,|\mathcal{M}_{m+1;(i,k)}^{(0)}(\{p\})|^2
$$

$$
+\mathcal{F}\!in\,[\mathrm{C}_i(y_{iQ};m)+\mathrm{CS}(m)]\,\boldsymbol{T}_i^2\,|\mathcal{M}_{m+1}^{(0)}(\{p\})|^2\Bigg]\Bigg\}. \tag{A.10}
$$

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